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Second Order Ordinary Differential Equation

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Abstract: Differential equations have proven to be an immensely successful instrument for modeling phenomena in science and technology. It is hardly an exaggeration to say that differential equations are used to define mathematical models in virtually all parts of the natural sciences. In this document, we will take the first steps towards learning how to deal with differential equations on a computer. This is a core issue in Computational Science and reaches far beyond what we can cover in this text. However, the ideas you will see here are reused in lots of advanced applications, so this document will hopefully provide useful introduction to a topic that you will probably encounter many times later. We will show you how to build programs for solving differential equations. More precisely, we will show how a differential equation can be formulated in a discrete manner suitable for analysis on a computer, and how to implement programs to compute the discrete solutions. The simplest differential equations can be solved analytically in the sense that you can write down an explicit formula for the solutions. However, differential equations arising in practical applications are usually rather complicated and thus have to be solved numerically on a computer. Therefore we focus on implementing numerical methods to solve the equations. The document Programming of ordinary differential equations describes more advanced implementation techniques aimed at making an easy-to-use toolbox for solving differential equations. Exercises in the present document and the mentioned document aim at solving a variety of differential equations arising in various disciplines of science.

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Introduction:

The study of differential equations originated in the investigation of laws that govern the physical world and were first solved by Sir Isaac Newton in seventeenth century (1642-1727) , who referred to them as 'fluxional equations.[1] The term differential equation was introduced by Gottfried Leibnitz, who was contemporary of Newton. [2] Both are credited with inventing the calculus. Many of the techniques for solving differential equations were known to mathematicians of this century, but a general theory for differential equations was developed by Augustin-Louis Cauchy (1789-1857). Applications to stock markets and problems related to finance and legal profession, studied towards the later part of twentieth century can be found in the work of Myron S. Scholes and Robert C. Merton who were awarded Nobel Prize of Economics in 1997. [3] The three aspects of the study of differential equations - theory, methodology and application - are treated in this book with the emphasis on methodology and application. The purpose of this chapter is two-fold: to introduce the basic terminology of differential equations and to examine how differential equations arise in endeavour to describe or model physical phenomena or real world problems in mathematical terms. [4]

Ordinary differential equations (ODEs) is a subject with a wide range of applications and the need of introducing it to students often arises in the last year of high school, as well as in the early stages of tertiary education. [5] The usual methods of solving second-order ODEs with constant coefficients, among others, rely upon the use of complex variable analysis, a topic to which the students in question may not have been previously introduced. The purpose of this article is to present an alternative approach in establishing the general solution for such types of equations without using complex numbers. [6]

The need for teaching differential equations and their applications arises—not only in mathematical courses, but also in subjects such as physics, engineering, mechanics, etc.—in the last year of senior high school, as well as in the early stages of higher education. In order for a sixth-former or a nonmathematics major college freshman to successfully attend an introductory differential equations course, he/she must at least have good knowledge of basic differentiation and integration techniques. A student at that level can usually cope with a first-order ordinary differential equation (ODE), which is either variable separable, homogeneous, exact or easily converted into an exact one, by applying the appropriate integrating factor. [7]

In this research paper we present a method in which we develop an analytic solution for certain classes of second-order differential equations with variable coefficients. By the use of transformations and by repeated iterated integration, we obtain a desired solution.[8] This represents an alternative way to obtain a solution to such methods as classic power series techniques and other approaches (see, for example, Ince 1, where the author uses successive approximations). It is, at times, more involved than traditional methods.[9]

DIFFERENTIAL EQUATIONS[10]

- An equation in which at least one term contains $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc.
- examples : $x + 4\frac{dy}{dx} = 5y$ $\rightarrow$ first order differential equation
  - it contains only first-order derivatives
  - it is called linear differential equation

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = x^2$$ $\rightarrow$ second order differential equation

- it contains a second order derivatives
- this is not a linear differential equation

Note :

- The order of a differential equation is the order of the highest derivative that it contains.
- Any differential equation represents a relationship between two variables, ***x*** and ***y***.

FIRST-ORDER DIFFERENTIAL EQUATIONS[11]

Type I : Separation of Variables

- is used when the equation is in the simplest first-order form of equation e.g.

$$h(y)\frac{dy}{dx} = g(x)$$ (Basic form)

To solve it :

1. Separate the variables *y* from *x*, i.e., by collecting on one side all terms involving *y* together with *dy*, while all terms involving *x* together with *dx* are put on the other side.
2. Integrate both sides.
3. If the solution can be defined explicitly, i.e., it can be solved for *y* as a function of *x*, then do it. If not, the solution can be defined implicitly, i.e., it cannot be solved for *y* as a function of *x*.

Examples. Solve the following differential equations:

1. $$y\frac{dy}{dx} = e^x$$

$$y\,dy = e^x\,dx \qquad (separate \;\; the \;\; variables)$$

$$\int y\,dy = \int e^x\,dx \qquad (integrate \;\; both \;\; sides)$$

$$\frac{y^2}{2} = e^x + C$$

$$y = \sqrt{2e^x + 2C} \quad or \quad y = \sqrt{2e^x + C_1}$$

2. $$(4y - \cos y)\frac{dy}{dx} = 3x^2$$

$$(4y - \cos y)dy = 3x^2\,dx$$

$$\int (4y - \cos y)\,dy = \int 3x^2\,dx$$

$$2y^2 - \sin y = x^3 + C$$

Note :

1. Solutions for the above examples are called the *general solutions*.
2. The general solution for example 1 is defined explicitly, where as, the general solution for example 2 is defined implicitly.
3. If *C* is to be evaluated, then the solution is called the *solution of the initial-value problem*.

Type II : Integrating Factors[12]

- this method is used when the equation is in the form of linear equation in which the variables cannot be separated, e.g. :

$$\frac{dy}{dx} + p(x)y = q(x)$$

(Basic form)

where *p(x)* and *q(x)* – continuous functions

- may or may not be constants

Some other examples :

1. $$\frac{dy}{dx} + x^2 y = e^x \qquad \Rightarrow p(x) = x^2 ; \quad q(x) = e^x$$

2. $$\frac{dy}{dx} + (\sin x)y + x^3 = 0 \Rightarrow p(x) = \sin x; \quad q(x) = -x^3$$

3. $$\frac{dy}{dx} + 5y = 2 \qquad \Rightarrow p(x) = 5; \quad q(x) = 2$$

Procedures to solve the differential equations using this method :

1. Must remember the basic form :

$$\frac{dy}{dx} + p(x)y = q(x)$$

2. Multiply both sides by μ gives :

$$\mu \frac{dy}{dx} + \mu p(x) y = \mu q(x)$$

where $\mu = e^{\int p(x)dx}$ - is called the integrating factor.
- this is what you have to calculate first
- can take the constant of integration to be zero at this step.

3. Integrate both sides of the equation obtained in (2) and the result obtained is :

$$\mu y = \int \mu q(x)\,dx$$

4. Finally, solve for *y*. Be sure to include the constant of integration in this step.

Examples : Solve the following differential equations :

1. $$\frac{dy}{dx} + 3y = e^{-3x}$$

$$Compared \quad with \quad the \quad basic \quad form, \quad p(x) = 3$$

$$so \quad \mu = e^{\int p(x)dx} = e^{\int 3dx} = e^{3x}$$

$$\therefore \frac{dy}{dx} + 3y = e^{-3x} \qquad \Rightarrow \mu y = \int \mu q(x)\,dx$$

$$e^{3x} y = \int e^{3x} e^{-3x}\,dx$$

$$e^{3x} y = \int dx$$

$$e^{3x} y = x + C$$

$$\therefore \quad y = \frac{x + C}{e^{3x}}$$

2. $$\frac{dy}{dx} = x - xy \quad and \quad y = 0 \quad when \quad x = 0$$

$$Basic \quad form: \ \frac{dy}{dx} + p(x)y = q(x)$$

$$\Rightarrow \frac{dy}{dx} + xy = x$$

$$\therefore \quad p(x) = x$$

$$\therefore \quad \mu = e^{\int p(x)dx} = e^{\int x dx} = e^{\frac{x^2}{2}}$$

$$\therefore \frac{dy}{dx} + xy = x \qquad \Rightarrow \mu y = \int \mu q(x)\,dx$$

$$e^{\frac{x^2}{2}} y = \int e^{\frac{x^2}{2}} x\,dx$$

$$e^{\frac{x^2}{2}} y = e^{\frac{x^2}{2}} + C$$

$$y = 0 \quad when \quad x = 0$$

$$\Rightarrow \quad C = -1$$

$$\therefore \quad y = \frac{e^{\frac{x^2}{2}} - 1}{e^{\frac{x^2}{2}}}$$

$$y = 1 - e^{-\frac{x^2}{2}}$$

SECOND-ORDER DIFFERENTIAL EQUATIONS[13]

Second-order linear differential equation has the following basic equation :

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

or in alternative notation,

$$y'' + p(x)y' + q(x)y = r(x)$$

where *p(x)*, *q(x)*, and *r(x)* are continuous functions.

If *r(x) = 0* for all *x*, then, the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$ is said to be *homogeneous*.

If $r(x) \neq 0$ for all *x*, then, the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$ is said to be *nonhomogeneous*.

Examples of second-order homogeneous differential equations :

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

$$y'' + e^x y = 0$$

Examples of second-order nonhomogeneous differential equations :

$$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - xy = e^x$$

$$y'' + y' - 3y = \sin x$$

Solving Second-Order Linear Homogeneous Differential Equations With Constant Coefficients

Two continuous functions *f* and *g* are said to be *linearly dependent* if one is a constant multiple of the other. If neither is a constant multiple of the other, then they are called *linearly independent*.

Examples:

f(x) = sin x and *g(x) = 3 sin x* => linearly dependent

f(x) = x and $g(x) = x^2$ => linearly independent

The following theorem is central to the study of second-order linear differential equations:

Theorem :

If $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are linearly independent solutions of the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \qquad (1)$$

then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad (2)$$

is the general solution of (1) in the sense that every solution of (1) can be obtained from (2) by choosing appropriate values for the arbitrary constants c_1 and c_2; conversely, (2) is a solution of (1) for all choices of c_1 and c_2.

At this level, we restrict our attention to second-order linear homogeneous differential equations with ***constant coefficients*** only, i.e.

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$ (basic form of equation)

where *p* and *q* are constants.

or I can rewrite :

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$ (basic form)

where *a*, *b*, and *c* are constants.

Replacing $\frac{d^2y}{dx^2}$ with m^2, $\frac{dy}{dx}$ with m^1, and y with m^0 will result

$$am^2 + bm + c = 0$$ => is called "*auxiliary quadratic equation*" or "*auxiliary equation*".

Thus, the general solution of the 2nd – order linear differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$ where *a*, *b*, and *c* are constants

depends on the roots of the auxiliary quadratic equation

$$am^2 + bm + c = 0$$ such that

i) if $b^2 - 4ac > 0$ (2 distinct real roots m_1 and m_2)

then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

ii) if $b^2 - 4ac = 0$ (1 real repeated root $m_1 = m_2 (= m)$)

then $y = c_1 e^{mx} + c_2 x e^{mx}$

iii) if $b^2 - 4ac < 0$ (2 complex roots $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$)

then $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Note :

Each of these solutions contains two arbitrary constants c_1 and c_2 since solution of 2nd – order differential involves 2 integrations.

Examples. Solve the following differential equations :

1. $$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = 0$$

$$\begin{aligned} Auxiliary\ \ eqn.\ :\ & 2m^2 + 5m - 3 = 0 \\ & (2m-1)(m+3) = 0 \\ & m = \frac{1}{2} \quad or \quad m = -3 \end{aligned}$$

$$\therefore\ the\ \ general\ \ solution,\ \ y = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x}$$

2. $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

$$\begin{aligned} Auxiliary\ \ eqn.\ :\ & m^2 - 4m + 4 = 0 \\ & (m-2)(m-2) = 0 \\ & m = 2 \end{aligned}$$

$$\therefore\ the\ \ general\ \ solution,\ \ y = c_1 e^{2x} + c_2 x e^{2x}$$

3. $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$$

$$Auxiliary \quad eqn. \; : \; m^2 - 2m + 4 = 0$$

$$m = \frac{2 \pm \sqrt{4-16}}{2}$$

$$m = 1 \pm \sqrt{3}i$$

$$\Rightarrow \; a = 1, \quad b = \sqrt{3}$$

$$\therefore \; the \; general \; solution, \; y = e^x(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$$

Solving Second-Order Linear Nonhomogeneous Differential Equations With Constant Coefficients

Now, we consider 2nd – order differential equation of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$$

where *p* and *q* are constants and *r(x)* is a continuous function.

Theorem :

The general solution of $y'' + py' + qy = r(x)$ is

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

where $c_1y_1(x) + c_2y_2(x)$ is the general solution of the homogeneous equation

$$y'' + py' + qy = 0$$

and $y_p(x)$ is any solution of $y'' + py' + qy = r(x)$.

In short, what is meant from the above theorem is that, the general solution for 2nd – order differential equation of the form $y'' + py' + qy = r(x)$ is

y = Complementary equation (C.E.) + Particular solution (P.S.)

where the C.E. is the solution of the differential equation of the form

$$y'' + py' + qy = 0 \qquad \text{(homogeneous)}$$

and the P.S. is any solution of the complete differential equation. It is determined by trial solution or initial guessing based on the form of *r(x)*.

Equation	Initial guess for y_p
$y'' + py' + qy = k$	$y_p = A$
$y'' + py' + qy = ke^{ax}$	$y_p = Ae^{ax}$
$y'' + py' + qy = a_0 + a_1x + \ldots + a_nx^n$	$y_p = A_0 + A_1x + \ldots + A_nx^n$
$y'' + py' + qy = a_1 \cos bx + a_2 \sin bx$	$y_p = A_1 \cos bx + A_2 \sin bx$

Modification rule : If any term in the initial guess is a solution of the C.E. , then the correct form for y_p is obtained by multiplying the initial guess by the smallest positive integer power of *x* required so that no term is a solution of the C.E.

In summary, a P.S. of an equation of the form $y'' + py' + qy = ke^{ax}$ **can be obtained as follows :**

Step 1. Start with $y_p = Ae^{ax}$ **as an initial guess.**

Step 2. Determine if the initial guess is a solution of the C.E.

$$y'' + py' + qy = 0.$$

Step 3. If the initial guess is not a solution of the C.E. , then

$y_p = Ae^{ax}$ is the correct form of a P.S.

Step 4. If the initial guess is a solution of the C.E., then multiply it by the smallest positive integer power of *x* required to produce a function that is not a solution of the C.E. . This will yield either $y_p = Axe^{ax}$ **or** $y_p = Ax^2e^{ax}$**.**

Examples. Solve the following differential equations :

1. $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 3$$

$$\Rightarrow \quad y = C.E + P.S.$$

$\underline{C.E.}$

$$Auxiliary \quad eqn. \ : \ m^2 - 6m + 5 = 0$$
$$(m-5)(m-1) = 0$$
$$m = 5 \quad or \quad m = 1$$

$\therefore$ *the general solution for* $C.E, \ y = c_1e^{5x} + c_2e^x$

$\underline{P.S.}$

$$r(x) = 3$$

$\therefore$ *the initial guess* : $y_p = A$

$$using \quad y = A$$

$$\frac{dy}{dx} = 0, \ \Rightarrow \ \frac{d^2y}{dx^2} = 0$$

substitute these values into the original eqn.

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 3$$

$$\Rightarrow \ 0 - 6(0) + 5A = 3$$

$$\therefore \quad A = \frac{3}{5}$$

$\therefore$ *the general solution for* $P.S., \ y_p = \frac{3}{5}$

Thus, the general solution is

$$y = c_1e^{5x} + c_2e^x + \frac{3}{5}$$

2. $2\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=\cos x-5\sin x$

$$\Rightarrow \quad y = C.E + P.S.$$

C.E.

$$Auxiliary \quad eqn. \; : \; 2m^2+2m+5=0$$

$$m=\frac{-2\pm\sqrt{-36}}{4}$$

$$m=-\frac{1}{2}\pm\frac{3}{2}i$$

$$\therefore \; the \; gen. \; sol. \; for \; C.E, \; y=e^{-\frac{1}{2}x}(c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x)$$

P.S.

$$r(x)=\cos x-5\sin x$$

$$\therefore \; the \; initial \; guess \; : \; y_p=A_1\cos x+A_2\sin x$$

$$using \quad y=A_1\cos x+A_2\sin x$$

$$\Rightarrow \quad \frac{dy}{dx}=-A_1\sin x+A_2\cos x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-A_1\cos x-A_2\sin x$$

substitute these values into the original eqn.

$$2\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=\cos x-5\sin x$$

$$\Rightarrow 2(-A_1\cos x-A_2\sin x)+2(-A_1\sin x+A_2\cos x)+$$
$$5(A_1\cos x+A_2\sin x)=\cos x-5\sin x$$

$$\Rightarrow \cos x(3A_1+2A_2)+\sin x(-2A_1+3A_2)=\cos x-5\sin x$$

compare coefficients of $\sin x$ *and* $\cos x$:

$$\begin{cases} 3A_1+2A_2=1 \\ -2A_1+3A_2=-5 \end{cases} \qquad \Rightarrow solve \; for \; A_1 \,\&\, A_2 \; yields$$

$$A_1=1, \qquad A_2=-1$$

$$\therefore \; the \; general \; solution \; for \; P.S., \; y_p=\cos x-\sin x$$

Thus, the general solution is

$$y=e^{-\frac{1}{2}x}(c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x)+\cos x-\sin x$$

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