On One to One Correspondence Mapping and Its Application To Linear Assignment Problem

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Abstract: One to one correspondence can be referred to as a relation between the elements of two sets, where every element of one set is paired with exactly one element of the other set, and every element of the other set is paired with exactly one element of the first set. There are no unpaired elements. In the same way, Assignment problem deal with assigning n items/activities/tasks etc to n machines/workers/contractors etc in the best way so that no one activity gets more than one machine/worker and no one worker get more than one activity assigned to it. In this paper one to one correspondence is examined with its applications to linear Assignment problem. A one to one correspondence algorithm for solving linear assignment problem is presented. Finally numerical problems were solved for both minimization and maximization assignment problems for further understanding, and it was observed that the optimal solution is obtained faster and easier than the existing methods.

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1. Introduction

Let assume that an office has one worker, and one task to be performed. How would you employ the worker? The immediate answer will be, the available worker will perform the task. Again, suppose there are two tasks and two workers are engaged at different rates to perform them, which worker should perform which task for maximum profit? Similarly, let there be *n* tasks available and n workers are engaged at different rates to perform them, which tasks should be given to which workers to ensure maximum efficiency. To answer the above, we must find, such an assignment by which the office gets maximum profit or minimum investment. Such problems are known as assignment problem. In the other words a one to one correspondence can be referred to as a mapping that is both one to one and onto, i.e., a bijective mapping of a finite set into itself.

The assignment problem like transportation problem is a special case of linear programming problem (LPP). It is concerned with one to one mapping, when *n* jobs are to be assigned to *n* facilities with a view to optimise the resource (s) required. The emphasis is on how assignment should be made in order to minimise the total cost involved or maximize the total value involved. It can be classified as balanced assignment problem in which the number of rolls (tasks/items/activities) equal to the number of columns (individual/machines/workers). Otherwise known as unbalanced assignment problem, if the problem is unbalanced, necessary numbers of dummy row (s)/column (s) are added such that the cost matrix is a square matrix. Assignment Problem is one of the fundamental combinatorial optimization problems, Panneerselvam (2009). In this work we intend to use one to one correspondence assignment algorithm to obtain the optimal assignment solution.

Mathematically an assignment is a bijective mapping of a finite set into itself, i.e., permutation. Every permutation ϕ of the set $N = 1, 2, \ldots, n$ corresponds in a unique way to a permutation matrix $X\phi = \left(x_{ij}\right)$ with $x_{ij} = 1$ for $j = \phi(i)\, and\, x_{ij} = 0, \text{ for } j \neq \phi(i)$, as in figure 1.1. Rainier and Eranda, (2011)

$$\phi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$X\phi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 1 \rightarrow 3 \\ 2 \rightarrow 2 \\ 3 \rightarrow 4 \\ 4 \rightarrow 1 \end{matrix}$$

Figure 1.1 representation of Assignment problem

The set of all assignments of *n* items will be denoted by S_n and has n! elements and can be described by the following equations called assignment constraints, (Panneerselvam 2009), Ruhul and Charles, (2008).

$$\left.\begin{aligned}&\sum_{i=1}^{n} x_{ij} = 1, j = 1,\ldots n\\&\sum_{j=1}^{n} x_{ij} = 1 \; for \;\; all \; i = 1,\ldots n\\&x_{ij} \in [0,1] \;\; for \;\; all \;\; i, j = 1,\ldots n\end{aligned}\right\} \quad (1)$$

Literature revealed that different methods exist for solving assignment problem such as Enumeration method, Simplex method, Transportation method and Hungarian method to mention a few. Out of all these, Hungarian method is the most efficient method. Hungarian method works on the principle of reducing the given cost matrix to the extent of having at least one zero in each row and column and then make optimal assignment, Ruhul and Charles (2008), Panverselvan (2009), and Kumal (2006). Basirzadeh (2012), propose a new approach for solving assignment problem namely Ones Assignment Method. This is based on reducing the given cost matrix to the extent of having at least one One in each row and column and then make optimal assignment in terms of these Ones. The method give optimal solution same as the effective Hungarian method. Also, Chadle and Muley (2013), revised Ones Assignment Method for solving assignment problem and the method was found effective.

In this paper a one to one correspondence algorithms for solving linear assignment problem is introduced, numerical illustration is given for both minimization and maximization problems for more understanding, and the optimal result was found faster and easier than the existing methods.

2. Material And Methods

2.1. Mathematical Formulation of The Problem

Given n tasks and n workers available with different skills, if the cost of performing j^{th} task by i^{th} worker is C_{ij} and x_{ij} denote the assignment of task j to worker i. Then, the Mathematical model for the assignment problem can be stated as

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Minimize

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad (2)$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1,$$

j^{th} *task will be performed by only one worker*

$$\sum_{i=1}^{n} x_{ij} = 1,$$

i^{th} *worker will perform only one task*

$$where \;\; x_{ij} = \begin{cases} 1 \; if \; i^{th} \text{ worker is assigned } j^{th} \text{ task} & (3) \\ 0 \; ; if \; i^{th} \text{ worker } is\; not \text{ assigned } j^{th} \text{ task} \end{cases}$$

2.2 A One to One Correspondence Algorithm For Solving Linear Assignment Problem

In this section, the algorithm for one to one correspondence is presented for solving linear assignment problem. Numerical illustrations follow it in the next section for more understanding of the method.

A one to one correspondence is a relation or mapping that is both one to one and onto. We shall states the following theorems:

Theorem 2.1

If all $c_{ij} \geq 0$ and there exist a solution $x_{ij} = X_{ij}$ such that

$$\sum_{i} c_{ij}x_{ij} = 0.$$

Then the solution is optimal.

Theorem 2.2

Let $\mathrm{F}: A \rightarrow B$ be linear, then F is one to one if and only if its maps linearly independent sets in A to linearly independent sets in B and F is onto if and only if it maps spanning sets of A to spanning sets of B.

Proof

Suppose that F is one to one and that S is a linearly independent set in A, we want to proof that $F(\mathrm{s}) = \{\mathrm{F}(s) : s \in S\}$ is also linearly independent.

$$c_1F(s_1) + c_2F(s_2) + \ldots + c_kF(s_k) = 0,$$

Let $for\; some\; s_i \in S, c_i \in G$

We have, $F(c_1s_1 + c_2s_2 + \ldots + c_ks_k) = 0$

and, $c_1s_1 + c_2s_2 + \ldots + c_ks_k \in N(F)$

Since F is one to one, so

$$N(F) = \{0\}, \Rightarrow c_1s_1 + c_2s_2 + \ldots + c_ks_k = 0,$$

hence, $F(s)$ is linearly independent.

Conversely let F maps linearly independent sets to linearly independent sets, let a be any non zero vector in A then {a} is linearly independent $\Rightarrow F(a) = 0, and\; N(F) = \{0\}$ and so F is one to one. Using the same procedure to prove the second part as follows;

Let F maps spanning sets to spanning sets and,

$$b \in B,$$

$$b = c_1F(s_1) + c_2F(s_2) + \ldots + c_kF(s_k) = 0,$$

$$for\ some\ s_i \in S, c_i \in G$$

$$b = F(c_1s_1 + c_2s_2 + \ldots + c_ks_k) \in R(F),$$

Hence, F is onto. Converse is also in the same way, and hence F is one to one and onto.

3. Steps For Solving Linear Assignment Problem (Using One To One Correspondence Algorithms)

This algorithm provides optimal solution to the linear minimization/ maximization assignment problem faster and easier. The steps are summarised as follows.

Step **1**: construct the data matrix for the assignment problem. Let row be the worker / resource and column as task / job / activity.

Step **2**: Write two columns, where column 1 represents resource / worker / person and column 2 represents task / job/ activity. Let A, B, C,..., Z denote workers / resource /person and I, II, III,.... represents tasks/ Job/ activities.

Step **3**: Find minimum for minimization problem (maximum for maximization problem) unit cost for each row, select it and write it in term of activities under column 2.

Step **4**: check, for each resource/ worker/ person if there is a unique activity /task /job, then assigned that activity /task uniquely to the corresponding resource /workers. Hence, optimal solution is obtained, otherwise go to step **5**.

Step **5**: select resource / worker / person with unique activity and assign that activity for the corresponding resource / worker. Delete the row

and its corresponding column for which resource has already been assigned.

Step **6**: Find the minimum (maximum) unit cost for the remaining rows, check if it satisfies step **5** then perform it, otherwise, check which rows have only one same task/ activity. Find the difference between minimum (maximum) and next minimum (maximum) unit cost.

Assign that activity which has maximum difference. Delete those rows and corresponding columns for which those resources have been assigned. In case there is tie in the difference for two and more than two activity then further take the difference between minimum (maximum) and next to next minimum (maximum) unit cost. And check which activity has maximum difference, assign that activity.

Step **7**: Repeat steps **4- 6** till all jobs are assigned uniquely to the corresponding activity.

Step **8**. If all jobs are assigned then calculate the total cost by using the formula.

$$\text{Total cost} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

4. Results

4.1. Numerical Application

In this section we consider some physical application for better understanding of unique mapping assignment method.

Problem 4.1

A business manager want to build a new warehouse, he want the task to be done in phases in the order of (I) foundation, (II)super structure, (III)roofing/ceiling, (IV)plumbing, (V)electrical, and (VI)finishes. He wants different contractors to handle different phases; the specification of materials was made generally to the contractors. Independently the contractors were asked to submit bills of quantities according to the categorization of phases. The table below shows the minimum cost in thousand per phase that each contractor accepted to carry out the contract if awarded to him. Determine the optimal contract assignment.

Table: 4.1

Contractor	Phase I ₦('000)	Phase II ₦('000)	Phase III ₦('000)	Phase IV ₦('000)	Phase V ₦('000)	Phase VI ₦('000)	Total ₦('000)
A	100	460	280	160	180	380	1560
B	200	160	220	460	500	250	1790
C	700	320	300	240	140	200	1900
D	320	460	420	140	200	340	1720
E	400	360	200	260	320	180	1720
F	320	420	230	240	240	300	1750

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Solution:

Consider table 4.1, select row A, where the minimum value is 100 representing phase I. Similarly, the minimum values for row B to row F are160, 140,140,180,230 respectively. As shown in Table 4.2 below.

Table 4.2

Contractors	Phases
A	I
B	II
C	V
D	IV
E	VI
F	III

From table 4.2, it is observed that different phases are meant for different contractors. Hence, we can assign phases uniquely to the contractors as in table 4.3.

Table 4.3

CT	*P.I*	*P.II*	*P.III*	*P.IV*	*P.V*	*P.VI*
A	$\boxed{100}$	460	280	160	180	380
B	200	$\boxed{160}$	220	460	500	250
C	700	320	300	240	$\boxed{140}$	200
D	320	460	420	$\boxed{140}$	200	340
E	400	360	200	260	320	$\boxed{180}$
F	320	420	$\boxed{230}$	240	240	300

Table 4.3 (*P* = *Phase*)

Therefore, one to one correspondence optimal solution is

Table 4.4

Contractors	Phases	Bills #'(000)	Time
A	I	100	
B	II	160	
C	V	230	
D	IV	140	
E	VI	140	
F	III	180	
Total	950		

Thus, the optimal assignment of contracts in phases to different contractors results to a total amount of ₦950,000 to complete the warehouse. This amount of money is less than each total amount of money per contractor.

Problem 4.2

A computer centre has five programmers, and five tasks to be performed. Programmers differ in efficiency and tasks differ in their intrinsic difficulty. Time in minutes each programmer would take to complete each task is given in effectiveness matrix. How should the tasks allocated so that the total production is maximized.

Table 4.3

Tasks	*Programmers*				
	I	*II*	*III*	*IV*	*V*
1	8	26	34	22	16
2	13	52	13	52	26
3	38	19	36	30	76
4	19	26	48	20	38
5	46	30	46	22	44

Solution:

Consider the data matrix in Table 4.5, select row 1 and select column III for which it has maximum value. In a similar way select rows 2, 3, 4, and 5 and select the respective column with maximum value as shown in table 4.6 below.

Table 4.6

Task	Programmers
1	III
2	II, IV
3	V
4	III
5	I, III

Table 4.7

Tasks	*Programmers*				
	I	*II*	*III*	*IV*	*V*
1	8	26	34	22	16
2	13	52	13	52	26
3	38	19	36	30	$\boxed{76}$
4	19	26	$\boxed{48}$	20	38
5	46	30	46	22	44

We are now left with tasks 1,2, and 5 and programmers I, II, and IV as shown in table 4.8below

Table 4.8

	I	*II*	*IV*
1	8	26	22
2	13	52	52
5	46	30	22

In this case, Task 3 has a unique programmer V, assign task 3 uniquely to programmer V as shown in Table 4.7, next delete row 3 and column V for which the subordinate has already been assigned. Also, since Task 1,4 and 5 has same programmer III, find the maximum production difference between maximum and next maximum for Task 1,4 and 5(bear in mind that row 3 and column V has been deleted). Here maximum production difference for task 1 is 8 (34 – 26), maximum production difference for Task 4 is 22 (48-26), while maximum production difference for task 5 is 16(46-22). Since 22 is the maximum difference represent Task 4, hence assign Task 4 to programmer III as shown in Table 4.7. Then delete row 4 and column III.

Again, select the maximum production for the remaining Tasks 1, 2 and 5 which is shown in Table 4.9.

Table 4.9

Task	Programmer
1	II
2	II, IV
5	I,

Since Task 5 has a unique Programmer, assign Task 5 uniquely to Programmer I as in Table 5.10 and delete row 5 and column I and we are left with Tasks 1,2 and programmer II and IV as in Table 5.11.

Table 4.10

	I	*II*	*IV*
1	8	26	22
2	13	52	52
5	$\boxed{46}$	30	22

Table 4.11

	II	*IV*
1	26	22
2	52	52

Again, since Task 1 and 2 have the same programmer II, the maximum differences between maximum and next maximum production for 1 and 2 are 4 and 0 respectively. Here 4 is the maximum so assign Task 1 uniquely to programmer II. Delete row 1 and column II.

At last Task 2 and programmer IV is left and hence, we assign Task 2 uniquely to programmer IV.

Finally, different Task, have assign programmer uniquely in a a one to one correspondence approach which is shown in Table 4.12.

Table 4.12

Task	programmer	Time
1	II	26
2	IV	52
3	V	76
4	III	48
5	I	46
	Total	248

The optimal solution is 248 minutes

Problem 4.3

A car hire company has one car at each of his five depots A, B, C, D, E. Customer requires a car in each town, namely I, II, III, IV, V. Distances (in km) between depots (origins) and towns (Destinations) are given in the distance matrix. How should a car be assigned as to minimize the distance travelled?

Table 4.13

	A	*B*	*C*	*D*	*E*
I	280	220	310	340	360
II	230	200	220	280	310
III	240	200	270	300	310
IV	60	130	60	130	180
V	70	100	30	120	170

Solution:

Consider the data matrix (Table 4.13). Select row I and column for which row I has a minimum value, the same way we select rows II to V as we have in Table 4.14.

Table 4.14

Column 1	Column 2
I	B
II	B
III	B
IV	A,C
V	C

There is no town with unique assignment, however town IV and V have same C, so we find the difference between minimum and the next minimum for town IV and V. The minimum difference for IV is 0 (60 -60) while the minimum difference for V is 40 (70 – 30). Since 40 is the maximum difference which represent town V and hence assign C uniquely to V as shown in 4.15. Next delete row V and column C. Also the minimum difference for I, II, and III are 60,30, 40 respectively. And since 60 is the maximum difference which represents I, hence assign I uniquely to B, and delete row I and column B as in Table 4.16.

Table 4.15

	A	*B*	*C*	*D*	*E*
I	280	$\boxed{220}$	310	340	360
II	230	200	220	280	310
III	240	200	270	300	310
IV	60	130	60	130	180
V	70	100	$\boxed{30}$	120	170

Table 4.16

	A	*D*	*E*
II	230	280	310
III	240	300	310
IV	60	130	180

Again select minimum value for the remaining II, III, IV as shown in Table 4.17 below

Table 4.17

Column 1	Column 2
II	A
III	A
IV	A

Table 4.18

	A	*D*	*E*
II	230	280	310
III	240	300	310
IV	$\boxed{60}$	130	180

There is no unique mapping, hence we find maximum difference between the minimum and the next minimum difference for II, III, and IV. The minimum differences are 50, 60, 70 respectively. Since 70 is the maximum difference which represent IV, then IV is uniquely assigned to A as shown in Table 4.18. Then delete row IV and column A.

Also, select the minimum value for the remaining II and III (since row IV and column A has been deleted) as in Table 4.19.

Table 4.19

Column 1	Column 2
II	D
III	D

There is no unique assign as II and III has same D. The minimum difference for II and III are 30 and 10 respectively. Since 30 is the maximum minimum difference which II. II is uniquely assigned to D as in Table 4.20. Delete row II and column D.

Table 4.20

	D	*E*
II	$\boxed{280}$	310
III	300	310

Lastly, row III and column E is left and hence, we assign row III uniquely to E.

Finally, different town have assigned car uniquely as we have in table 4.21.

Table 4.21

Town	Depot	Distance
I	B	220
II	D	280
III	E	310
IV	A	60
V	C	30
Total		900

Thus the optimal solution is obtained.

6. Conclusion.

In this work, one to one correspondence algorithm is presented for solving linear assignment problems. Numerical problems were solved for both minimization and maximization assignment problem and the result obtained shows that this method gives an optimal result and faster compared to the existing methods. It is valid for both minimization and maximization problem.

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