

Tiling of Chaotic Manifold and its Fractal folding

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Abstract: In this paper we introduce the notation of tiling chaotic manifold. We also study some geometric characters on chaotic manifold. The fractal folding of chaotic manifold is discussed. The fractal tiling, is a tiling which possesses self-similarity and the boundary of which is a fractal, is presented. Some applications in real life on chaos theory are achieved.

[F. Salama, H. Rafat. **Tiling of Chaotic Manifold and its Fractal folding**. *J Am Sci* 2012;8(12):198-202]. (ISSN: 1545-1003). <http://www.americanscience.org>.30

Mathematics Subjected Classification (2010): 51H20, 51H30, 57M50, 28A80, 65P99.

Keywords: Chaotic Manifold, Tiling, Fractal folding, retraction.

1. Introduction

The word 'chaos' takes on many connotations. Mathematicians, however, have narrowed the meaning of the word to explain "the reason for unpredictable behavior". Then how does chaos theory fit in? "Chaos theory attempts to explain the fact that complex and unpredictable results that can and will occur in systems that are sensitive to their initial conditions". Andrew Ho dispels some "misconceptions" about the meaning of chaos theory: "Chaos theory is not about disorder . . . the 'chaos' in chaos theory is order—not simply order, but the very essence of order". To illustrate this point, shapes scientists described as "grainy, hydra like, in between, pimply, . . . sea weedy, . . ." can now be described in a mathematical way. This new branch of mathematics is still in its infancy, though. For example, the motion of pool balls can be predicted accurately only a few seconds into the future, because small "imperfections" begin to affect the ball immediately. Several principles in chaos theory exist. Two of the most important are that complex systems have an underlying order, and that simple systems can produce complex behavior. Fundamental components of chaos theory are fractals, imaged sets of numbers that always resemble the original image when magnified, area. A tree, for example, is nothing more than miniscule copies of itself. A large branch has smaller branches that are similar to the large branch. The smaller branches are composed of twigs, the twigs composed of even smaller parts. The twig is a replica of the smaller branch, and the large branch is a replica of the entire tree. Chaos theory is utilized in computer systems for creating virtual worlds. Fractals are helpful in creating realistic clouds, rocks, and shadows. Fractals are also effectively used in

image compression and movie special effects. Texas Instruments used chaos theory in a project to make a new chipset for the conversion of analog signals to their digital counterpart. To create the new chip, Texas Instruments' engineers modeled simple genes: "If gene A is active and gene B is not, then activate gene C". This created a small amount of complexity, enough to make the new chipset more effective than predecessors. A system is called chaotic if it is impossible to make accurate long-term predictions about the behavior of the system. One of the first researchers in chaos theory was a meteorologist, Edward Lorenz. He was using a set of equations to model the weather. When he solved the system numerically, he found that a particle moving subject to atmospheric forces has a very complicated trajectory. He also found that any such a trajectory always approaches the same general pattern of an attractor. This pattern is butterfly shaped and is now known as the Lorenz attractor. The conclusion of these observations is that a very small change in initial conditions can produce unpredictable and sometimes drastic results by triggering a series of increasingly significant events. In theory, the flutter of a butterfly's wings in Australia could, for example, produce a snowstorm in the Northeastern, thousands of miles away. Chaos theory is being applied to many real world situations in nature, at work, in financial markets, and in computer technology. Toby Tenenbaum argues for "nonlinear work [environments]." He says workers are more independent; they can work anytime. A chaotic organizational method would be more effective than the traditional self-organizing method. He feels that chaos in management can be beneficial to an organization, creating a "nonlinear world of work".

Mathematicians and physicists have entered Wall Street because brokers are interested in creating a model of the stock market. The purpose of the model is to predict crashes and other unforeseen risk investors are taking. The difficulty of creating a workable model is in the form of a paradox. In the short term, the market seems to behave randomly, but long term behavior seems "deterministic." However, it has been resolved that this paradox can exist because the long-term result is built on random short-term results [5, 6, 14, 17 and 23].

Benoit Mandelbrot coined fractal from the Latin adjective fractus. The corresponding Latin verb frangere means, "to break": to create irregular fragments. It is therefore sensible-and how appropriate for our needs! -That, in addition to "fragmented", fractus should also mean "irregular," both meanings being preserved in fragment. This means that a "fractal" is a rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced/size copy of the whole. Mathematically, a "fractal" is a set of points whose fractal dimension exceeds its topological dimension. In other words, a "fractal" is a geometrical figure that has the following properties:

- The geometrical figure has self-similarity,
- The geometrical figure has a dimension that is not an integer.

Fractals appear to be more popular in the status quo for their aesthetic nature than they are for their mathematics. Everyone who has seen a fractal has admired the beauty of a colorful, fascinating image, but what is the formula that makes up this glitzy image? The classical "Euclidean geometry" that one learns in school is quite different than the "fractal geometry" mainly because fractal geometry concerns nonlinear, nonintegral systems while Euclidean geometry is mainly oriented around linear, integral systems. Hence, Euclidean geometry is a description of lines, ellipses, circles, etc. However, fractal geometry is a description of algorithms. There are two basic properties that constitute a fractal. First, is self-similarity, which is to say that most magnified images of fractals are essentially indistinguishable from the unmagnified version. A fractal shape will look almost, or even exactly, the same no matter what size it is viewed at. This repetitive pattern gives fractals their aesthetic nature. Second, fractals have non-integer dimensions. This means that they are entirely different from the graphs of lines and conic sections that we have learned about in fundamental Euclidean geometry classes [1, 4, 8, 11, 12, and 15].

A tiling of spherical, Euclidean or hyperbolic plane is a collection τ of polygons, called tiles, that completely cover the plane without overlaps or gaps.

The sides or edges of the tiles are called the edges of the tiling and the vertices of the tiles are the vertices of the tiling. The art of tilings and patterns can be found in the history of all ancient civilizations. However, the mathematical theory of tilings is comparatively recent and many aspects of the subject are still unexplored. New constructions of tilings are discovered from time to time. For instance, in the 1960, sets of prototiles were discovered which admit infinitely many tilings of the plane but those tilings were all non periodic, i.e. they did not possess translational symmetry. The Penrose tilings are remarkable examples which have local five-fold rotational symmetry but with no translational repetition [7].

2. Definitions

1. A C^k n -dimensional manifold M is a non-empty (second-countable, Hausdorff) topological space such that:

(a) M is the union of open subsets U_α , and each U_α is equipped with a homeomorphism φ_α , taking U_α to an open set in R^n , i.e. $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha) \subset R^n$

(b) If $U_\alpha \cap U_\beta \neq \emptyset$, then the overlap map $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map (See Fig. (1)).

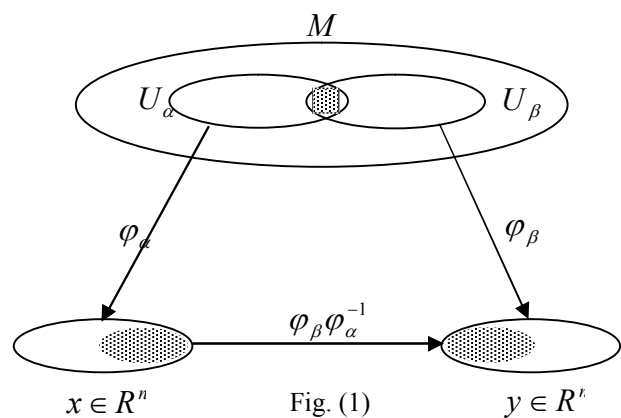


Fig. (1)

Each pair $(U_\alpha, \varphi_\alpha)$ is called a "chart" on M and the collection $A = \{(U_\alpha, \varphi_\alpha)\}$ of all charts is called a "(smooth) atlas" on M . The space M taken together with atlas A will be called a "smooth manifold of dimension n " or "smooth n -manifold" or " C^∞ n -manifold".

Moreover, if the topological space M satisfies condition (a) only the manifold will be called a “*topological manifold*” or simply a “*manifold*”[2].

2. The field of folding began with S. A. Robertson’s work, in 1977, on isometric folding of a Riemannian manifold M into another N , which send any piecewise geodesic path in M to a piecewise geodesic path with the same length in N [22]. More studies on the folding of manifolds are studied in[3, 9, 10, and 18-20].
3. A subset A of a topological space X is called a “retract” of X if there exists a continuous map $r : X \rightarrow A$ (called a retraction) such that $r(a) = a \quad \forall a \in A$, where A is closed and X is open [13, and 16].
4. The chaotic manifold is a manifold changed by the time into homeomorphic manifolds either with fixed point $P_i, i = 1,2,\dots,n$, see Fig.(2), or with no-fixed point , see Fig.(3).[21]

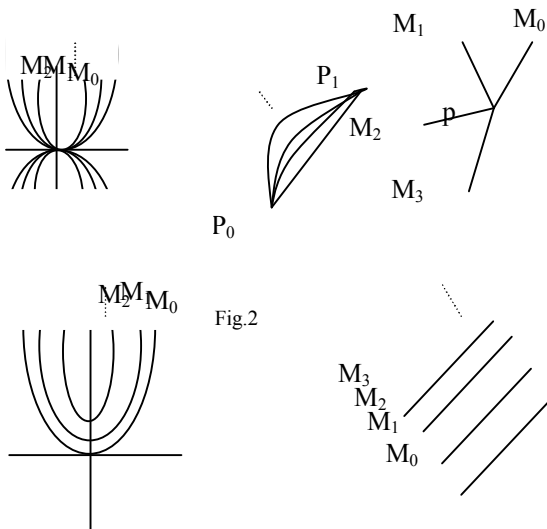


Fig.2

3. Main results

Aiming to our study we will introduce the following:
 The tiling of chaotic manifold which has n -physical character is a tiling for $n + 1$ manifolds $T_{0h}, T_{1h}, T_{2h}, \dots, T_{nh}$, where T_{0h} is the tiling for the geometric manifold, where T_{1h} is the tiling of the first chaotic manifold, where T_{2h} is the tiling of the second chaotic manifold, ..., and where T_{nh} is the tiling of n - chaotic manifold.

We will discuss the types of tiling of chaotic manifolds:

If we consider a chaotic manifold M , see Fig.(4).

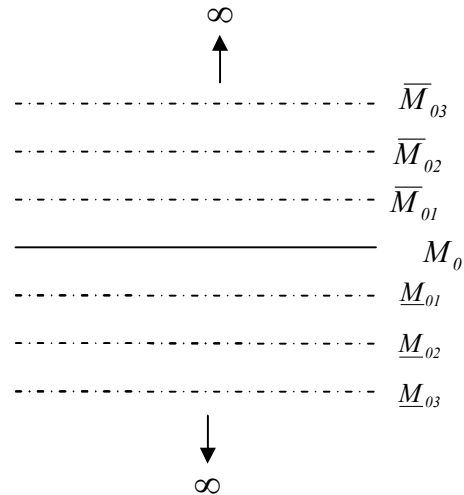


Fig.(4)

Where, M_0 is the geometric manifold and $\overline{M}_{0i}, \underline{M}_{0i}$ are the chaotic manifolds, on the geometric manifold, which have some physical characters but this physical characters on the chaotic manifolds may be the same or different characters. The tiling of chaotic manifold are based on two variables the first one is the physical characters (same or different) and the second one depend on the closed and open of the chaotic manifold (geometric or chaotic).

Therefore, we have four cases:

Case1: If the chaotic manifold M , in which the geometric manifold is closed and the chaotic are open, then by the definition of tiling any tiling of geometric manifold does not implies to tiling of chaotic one, see Fig.(5)

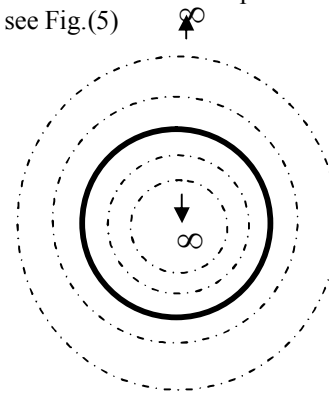


Fig.(5)

Remark: The chaotic manifold is open or closed according to the physical characters.

Case2: If the geometric and chaotic are closed, then tiling of M_0 implies to tiling of \overline{M}_{0i} and \underline{M}_{0i} . Also,

we can do the tiling for any chaotic manifold \bar{M}_{0i} or \underline{M}_{0i} without doing tiling to the geometric manifold.

Case3: If the geometric manifold is open and the chaotic is closed, then tiling of \bar{M}_{0i} and \underline{M}_{0i} does not implies to tiling of M_{0i} .

Case4: If the geometric and chaotic are open , then we cannot doing any tiling for geometric and chaotic manifolds.

From the above discussion we will arrive to the following theorem.

Theorem1. *Under tiling of chaotic manifold the local variation of the chaotic manifold is in the sectional curvature only.*

Now we will discuss the geometric transformation on the chaotic manifold and the relations between them.

We will define a type of folding:

$$f: (x_1, x_2) \rightarrow (x_1, Ax_2), \quad A \leq 1.$$

The fractal folding of a chaotic manifold is defined by:

$$\begin{aligned} f_{0i}: M_{0i}^{n+1/p} &\rightarrow M_{0i}^{n+1/p} \\ f_0(x_1, x_2, \dots, x_n, \varepsilon_1 x_{n+1}) &= (x_1, x_2, \dots, x_n, \varepsilon_1 |x_{n+1}|), \quad |\varepsilon_1| < 1. \\ f_{01}: M_{0i}^{n+1/p} &\rightarrow M_{0i}^{n+1/p} \\ f_{01}(x_1, x_2, \dots, x_n, \varepsilon_1 x_{n+1}) &= (x_1, x_2, \dots, x_n, \varepsilon_2 |x_{n+1}|), \quad \varepsilon_2 < \varepsilon_1 \\ f_{02}(x_1, x_2, \dots, x_n, \varepsilon_2 x_{n+1}) &= (x_1, x_2, \dots, x_n, \varepsilon_3 |x_{n+1}|), \quad \varepsilon_3 < \varepsilon_2 \\ &\vdots \\ f_{0n}(x_1, x_2, \dots, x_n, \varepsilon_n x_{n+1}) &= (x_1, x_2, \dots, x_n, \varepsilon_{n+1} |x_{n+1}|), \quad \varepsilon_{n+1} \rightarrow 0 \end{aligned}$$

Then,

$$\lim_{n \rightarrow \infty} f_{0n} \left(M_{1k}^{n+1/p} \right) = M_{1k}^n$$

There are a corresponding induced sequences of folding

$$\begin{aligned} f_{1k}: M_{1k}^{n+1/p} &\rightarrow M_{1k}^{n+1/p} \\ f_{2k}: M_{2k}^{n+1/p} &\rightarrow M_{2k}^{n+1/p} \\ &\vdots \\ f_{\infty k}: M_{\infty k}^{n+1/p} &\rightarrow M_{\infty k}^{n+1/p}, \\ f_{1k1}, f_{1k2}, \dots, f_{1kn}, \\ f_{2k1}, f_{2k2}, \dots, f_{2kn}, \\ &\vdots \\ f_{mk1}, f_{mk2}, \dots, f_{mkn} \end{aligned}$$

Such that

$$\lim_{n \rightarrow \infty} f_{1kn} \left(M_{1k}^{n+1/p} \right) = M_{1k}^n,$$

$$\lim_{n \rightarrow \infty} f_{2kn} \left(M_{2k}^{n+1/p} \right) = M_{2k}^n,$$

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$$\lim_{n \rightarrow \infty} f_{mkn} \left(M_{mk}^{n+1/p} \right) = M_{mk}^n$$

Thus, the following theorems are obtained.

Theorem 2. *Any fractal folding of the geometric manifold M_{0k} induces fractal folding of every pure chaotic manifold.*

Theorem 3. *The end of the limit of fractal folding of chaotic manifold is equal to the end of its retractions.*

Theorem 4. *The fractal retraction of the geometric chaotic manifold induces a fractal dimensional of the pure chaotic manifold but the inverse is not true.*

Finally, we discuss the fractal tiling of chaotic manifold. Fractals and tiling are two fields in mathematics that are generally thought of as separate. However, they can be combined to form a variety of visually appealing constructs that possess fractal character and at the same time obey many of the properties of tiling. Since a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. The constructs described in this paper do not cover the entire Euclidean plane, however, they do obey the restrictions on gaps and overlaps. In the chaotic manifold each tile is a closed topological disc. The fractal tilings are also edge to edge unless otherwise noted, i.e. the corners and sides of the tiles coincide with the vertices and edges of the tilings. However, they are not well behaved by the criteria of normal tilings in one particular, namely they contain singular points, defined as follows: every circular disc, however small centered at a singular points meets an infinite number of tiles. Since any fractal tiling of the general sort described here will contains singular points, this will be not be considered a property that presents a fractal tiling from being described as well-behaved.

Thus the fractal tiling share the following properties:

- Each fractal tiling is constructed from a single prototile, to which all the tiles are similar.
- The prototiles have edges of two lengths, denoted “long” and “short”.
- The fractal tiling is by design edge to edge. As a consequence of this choice, from a given generation of tiles to the next smaller generation, the tiles are scaled by the ratio of the short to long edges of the prototile.

- d. The fractal tiling is bounded in the Euclidean plane.

4. Some applications of chaos theory in real life

Chaos even has applications outside of science. Computer art has become more realistic through the use of chaos and fractals. Now, with a simple formula, a computer can create a beautiful and realistic tree. Instead of following a regular pattern, the bark of a tree can be created according to a formula that almost, but not quite, repeats itself. Chaos has already had a lasting effect on science, yet there is much still left to be discovered. Many scientists believe that twentieth century science will be known for only three theories: relativity, quantum mechanics, and chaos. Aspects of chaos show up everywhere around the world, from the currents of the ocean and the flow of blood through fractal blood vessels to the branches of trees and the effects of turbulence. Chaos has inescapably become part of modern science. As chaos changed from a little-known theory to a full science of its own, it has received widespread publicity. Chaos theory has changed the direction of science: in the eyes of the general public, physics is no longer simply the study of subatomic particles in a billion-dollar particle accelerator, but the study of chaotic systems and how they work. Finally, understanding chaos understands life.

Acknowledgements:

The authors are deeply indebted to the team work at the deanship of the scientific research—Taibah University for their valuable help and critical guidance and for facilitating many administrative procedures. This research work was financed _supported_ by Grant no_1072/1433_ from the deanship of the scientific research at Taibah University, Al-Madinah Al-Munawwarah, Saudi Arabia.

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11/5/2012