Dual Construction of Developable Ruled Surface

*Nassar H. Abdel-All, R.A.Huesien, and Ali Abdela Ali

Mathematics Department, Faculty of Science, Assiut University Assiut 71516, Egypt
*nhabdeal2002@yahoo.com

Abstract: In this paper, some of developable ruled surfaces are constructed using the dual representation of plane curves through a dual unit space curve on the dual unite sphere. These surfaces are studied and plotted.
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1. Introduction:

Dual numbers were introduced in the 19th century by Clifford, and there application to rigid body kinematics was subsequently generalized by Kafelnikov and study in their principle of transference. The principle of transference states that if dual numbers replace real ones then all relations of vector algebra for intersecting lines are also valid for skew lines .In practice, this means that all rules of vector algebra for the kinematics of a rigid body with a fixed point (spherical kinematics) also hold for motor algebra of a free rigid body (spatial kinematics). As a result, a general rigid body motion can be described by only three dual equation rather than six real ones. For several decades there were attempts to apply dual numbers to rigid body dynamics Investigators showed that the momentum of a rigid body can be described as a motor that obeys the motor transformation rule; hence, its derivative with respect to time yields the dual force. However in those investigations, while going from the velocity motor to the momentum motor, there was always a need to expand the equation to six dimension and to treat the velocity motor as two separate real vectors. This process actually diminishes one of the main advantages of dual numbers namely, compactness of representation. Screws in the space can be represented by dual vectors at the origin. The components of a dual vector consisting of a line vector at the origin and the perpendicular moment vector for the line vector in the space are equal to Plückers line coordinates. Furthermore the space of lines could be represented by points on the unit sphere and points in the tangential planes affiliated to each point on the sphere [3].

1.1 Dual Numbers [2],[3],[4]

A dual number $\hat{x}$ is defined as an ordered pair of real numbers (x, x^*) expressed formally as

$$\hat{x} = x + \varepsilon x^*. \tag{1.1}$$

where x is referred to as the real part and x^* as the dual part of The symbol ε is a multiplier which has the property $\varepsilon^2 = 0$. The algebra of dual numbers results from (1.1). Two dual numbers are equal if and only if their real and dual parts are equal, respectively. As in the case of complex numbers, addition of two dual numbers is defined as

$$(x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon(x^* + y^*) \tag{1.2}$$

Multiplication of two dual numbers result in:

$$(x + \varepsilon x^*)(y + \varepsilon y^*) = xy + \varepsilon(x^* y + x y^*) \tag{1.3}$$

Division of dual numbers $\frac{\hat{x}}{\hat{y}}$ is defined as

$$\frac{\hat{x}}{\hat{y}} = \frac{x}{y} + \varepsilon\left(\frac{x^*}{y} - \frac{xy^*}{y}\right), y \neq 0 \tag{1.4}$$

Remark that: division is possible and unambiguous only if $y \neq 0$.

1.2 Dual function

Dual function of dual number presents a mapping of a space of dual numbers on itself, namely.

$$\hat{f}(\hat{x}) = f(x, x^*) + \varepsilon f^*(x, x^*) \tag{1.5}$$

where $\hat{x} = x + \varepsilon x^*$ is a dual variable, f and f^* are two ,generally different, functions of two variables. The dual function (1.5) is said to be analytic if it satisfies the following

$$\hat{f}(x + \varepsilon x^*) = f + \varepsilon f^* = f(x) + \varepsilon(x^* f'(x) + \tilde{f}(x)), \quad ' = \tfrac{d}{dx} \tag{1.6}$$

where $\tilde{f}(x)$ is an arbitrary function of a real part of a dual variable. The analytic condition for dual function is

$$\frac{\partial f^*}{\partial x^*} = \frac{\partial f}{\partial x} \qquad (1.7)$$

The derivative of such a dual function with respect to a dual variable is

$$\frac{d\hat{f}(\hat{x})}{d\hat{x}} = \frac{\partial f^*}{\partial x^*} + \varepsilon \frac{\partial f^*}{\partial x} \qquad (1.8)$$

Taking into account (1.7) we have:

$$\frac{d\hat{f}(\hat{x})}{d\hat{x}} = \frac{\partial f}{\partial x} + \varepsilon \frac{\partial f^*}{\partial x} = f'(x) + \varepsilon(x^* f''(x) + \tilde{f}'(x)) \qquad (1.9)$$

If a function $f(x)$ has the derivative $f'(x)$, its value for the dual argument $\hat{x} = x + \varepsilon x^*$ is denoted by $\hat{f}(\hat{x})$ or $\hat{f}(x)$. Using the formal Taylor expansion of the function $\hat{f}(x)$ with the property $\varepsilon^2 = 0$; we have

$$\hat{f}(x) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x) \qquad (1.10)$$

1.3 Dual space

The set

$$D^3 = D \times D \times D = \{\hat{x} : \hat{x} = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*)$$
$$= (x_1, x_2, x_3) + \varepsilon(x_1^*, x_2^*, x_3^*) = \underline{x} + \varepsilon \underline{x}^*, \ \underline{x}, \underline{x}^* \in R^3$$

is a module on the ring D and is called the dual space (vector space defined on the field of dual numbers). For any $\hat{\underline{x}} = \underline{x} + \varepsilon \underline{x}^*$, $\hat{\underline{y}} = \underline{y} + \varepsilon \underline{y}^*$, the scalar or inner product and the vector product $\langle \hat{\underline{x}}, \hat{\underline{y}} \rangle, \hat{\underline{x}} \wedge \hat{\underline{y}}$ of $\hat{\underline{x}}$ and $\hat{\underline{y}}$ by are defined by, respectively

$$\langle \hat{\underline{x}}, \hat{\underline{y}} \rangle = \langle \underline{x}, \underline{y} \rangle + \varepsilon(\langle \underline{x}, \underline{y}^* \rangle + \langle \underline{x}^*, \underline{y} \rangle) \qquad (1.11)$$

$$\hat{\underline{x}} \wedge \hat{\underline{y}} = (\underline{x}_2 \underline{y}_3 - \underline{x}_3 \underline{y}_2, \underline{x}_3 \underline{y}_1 - \underline{x}_1 \underline{y}_3, \underline{x}_1 \underline{y}_2 - \underline{x}_2 \underline{y}_1) \qquad (1.12)$$

Where

$$\hat{\underline{x}} = (y_i + \varepsilon x_i^*), \ \hat{\underline{y}} = (y_i + \varepsilon y_i^*) \in D^3, 1 \le i \le 3$$

If $x \neq 0$, The norm $\|\hat{x}\|$ of is defined by

$$\|\hat{\underline{x}}\| = \sqrt{\langle \hat{\underline{x}}, \hat{\underline{x}} \rangle} = \|\underline{x}\| + \varepsilon \frac{\langle \underline{x}, \underline{x}^* \rangle}{\|\underline{x}\|} \qquad (1.13)$$

A dual vector $\hat{\underline{x}} = \underline{x} + \varepsilon \underline{x}^*$ is a dual unite vector if it satisfies the following $\langle \underline{x}, \underline{x}^* \rangle = 0$ and $\langle \underline{x}, \underline{x} \rangle = 1$. Then we have that $\langle \hat{\underline{x}}, \hat{\underline{x}} \rangle = 1$. Then a dual vector $\hat{\underline{x}}$ with unit norm is called a dual unit vector The subset

$$S^2 = \{\hat{\underline{x}} = \underline{x} + \varepsilon \underline{x}^* : \|\hat{\underline{x}}\| = (1,0); \underline{x}, \underline{x}^* \in R^3\} \subset D^3$$

is called the dual unit sphere with the center $\hat{O}$ in D^3.

1.4 Dual space curve

If every $x_i(u)$ and $x_i^*(u)$, $1 \le i \le n$ and $u \in R^n$, are differentiable dual valued functions, the dual vector field $\hat{x}(u)$ is defined as the following $\hat{x} : u \subset R^n \to D^n$, $\hat{x}(u) = x(u) + \varepsilon x^*(u)$.
Dual space curve is a dual vector field of one variable defined as the following $\hat{x} : I \subset R \to D^3$, where

$$u \to \hat{x}(u) = (x_1(u) + \varepsilon x_1^*(u), x_2(u) + \varepsilon x_2^*(u), x_3(u) + \varepsilon x_3^*(u)) = x + \varepsilon x^* \in D^3$$

is differentiable. The real part $\underline{x}(u)$ is called the indicatrix of the space curve $\hat{x}(u)$. The dual arc length of the curve $\hat{x}(u)$ from u_1 to u is defined as

$$\int_{u_1}^{u} \|(\hat{\underline{x}})'\| du = \int_{u_1}^{u} \|(\underline{x})'\| du + \varepsilon \int_{u_1}^{u} \langle \underline{T}, (\underline{x}^*(u))' \rangle du = s + \varepsilon s^*$$

where $\underline{T}$ is a unit tangent vector of $\underline{x}(u)$. From now on we will take the arc length s of $\underline{x}(u)$ as the parameter instead of *u*.

Dual Developable Ruled surface

Consider a dual unite space curve which is given by the following equation

$$\hat{x}(u^1) = \hat{x}(u^1, \varepsilon) = x(u^1) + \varepsilon x^*(u^1) \qquad (2.1)$$

then we have the following

$$\langle \underline{x}, \underline{x} \rangle = 1, \ \langle \underline{x}, \underline{x}^* \rangle = 0, \ \langle \underline{x}, \underline{x}' \rangle = 0, \ \langle x^{*\prime}, x' \rangle = 0 \qquad (2.2)$$

From these relations, we have $x^* \times x' = 0$.

$$\sigma : R(u^\alpha) = R(u^1, u^2) = x(u^1) \times x^*(u^1) + u^2 x(u^1) \qquad (2.3)$$

Consider a ruled surface defined by the vector equation
The tangent vectors to the surface σ are given as

$$R_1 = x \times x^{*\prime} + u^2 x', \quad R_2 = x, \quad R_\alpha = \frac{\partial R}{\partial u^\alpha}, \quad ' = \frac{d}{du^1} \tag{2.4}$$

Then the coefficients of the first fundamental form are given as

$$g_{11} = \langle R_1, R_1 \rangle = \|x \times x^{*\prime}\| + u^2 x'.(x \times x^{*\prime}) + (u^2)^2 \|x'\| \tag{2.5}$$

$$g_{12} = \langle R_1, R_2 \rangle = 0, \quad g_{22} = \langle R_2, R_2 \rangle = 1 \tag{2.6}$$

Thus the normal vector field is defined as

$$N = \frac{R_1 \times R_2}{g} = \frac{\langle x', x^* \rangle x + x^{*\prime} + u^2(x' \times x)}{g}, \quad g = det(g_{\alpha\beta}) \tag{2.7}$$

Also the 2nd derivatives are

$$R_{11} = x' \times x^{*\prime} + x \times x^{*\prime\prime} + u^2 x'' \quad R_{12} = x' \quad R_{22} = 0 \tag{2.8}$$

Then the coefficients of the second fundamental forms are defined as

$$L_{22} = \langle N, R_{22} \rangle = 0, \quad L_{12} = 0 \tag{2.9}$$

The Gaussian curvature given from the following equation

$$K = \frac{L_{22} L_{11} - L^2{}_{12}}{g} \tag{2.10}$$

Thus from (2.9) we have $K = 0$ and the ruled surface (2.3) is a type of developable ruled surface [8].

Spherical Image of plane Curve

Here we gave a general method to construct the spherical image of a plane curve. There for we considerplane curve $C : \underline{X} = (x(u^1), y(u^1)) \subset R^2$ and a unite sphere S^2 given as $S^2 : x^2 + y^2 + (z-1)^2 = 1$ .Let (a,b,0) be any point on the plane curve C. Hence we may write the straight line passing through the north pole (0,0,2) of the sphere S^2 and (a,b,0) by the normal equations

$$\ell : \frac{x}{a} = \frac{y}{b} = \frac{2-z}{2} \tag{3.1}$$

Let $(\alpha, \beta, \gamma) \in S^2$ be the point of intersection of the line l and the unite sphere S^2 without the north pole point $(0,0,2) \in S^2$. For this point, the equation (3.1) becomes

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{2-\gamma}{2} \tag{3.2}$$

Then we have

$$a = \frac{2\alpha}{2-\gamma}, \quad b = \frac{2\beta}{2-\gamma} \tag{3.3}$$

And from the equation of the sphere S^2, it is easy to see that

$$\alpha = \frac{4a}{a^2+b^2+4} \quad \beta = \frac{4b}{a^2+b^2+4} \quad \gamma = \frac{4a}{a^2+b^2+4} \tag{3.4}$$

Thus we have one to one correspondence $C \subset R^2 \to \varsigma = (\alpha, \beta, \gamma) \subset S^2 \subset R^3$.The curve $\varsigma = (\alpha, \beta, \gamma)$ is called the spherical image of the plane curve C and this correspondence is called stereographic projection [1],[2]. For the spherical curve $\varsigma = (\alpha, \beta, \gamma)$, its possible to obtain a space curve

$$r = r(u^1) = (\alpha, \beta, \eta) = (x_1, x_2, x_3), \quad \eta = \gamma - 1 \tag{3.5}$$

parameterized by an arc length (unit speed curve) which has the same trace as ς. In fact let $s = s(u^1) = \int_0^{u^1} \|\varsigma'(u^1)\| du^1, \quad u^1 \in$ the domain of the curve ς.Since $\frac{ds}{du^1} = \|\alpha'(u^1)\| \neq 0$,the function $s = s(u^1)$ has a differentiable inverse function s^{-1} of s. Now set $r = \varsigma(u^1)$. u^1 and Clearly, we have $\|r'(s)\| = \left\|\varsigma'(u^1).\frac{du^1}{ds}\right\| = 1$. This show that r has the same trace as ς and is parameterized by arc length. It is usual to say that *r* is a parameterizations of ς by arc length. This fact allows us to extend all local concepts defined to the curve ς. Thus, we say that the curvature $k(u^1)$ of ς at u^1 is the curvature of the parameterized curve *r* of ς by arc length $s = s(u^1)$ at the corresponding point. This is clearly independent of the choice of *r*, it is often convenient to use u^1 as a parameter instead of s in the reparameterized curve *r*.

Let $r(u^{1)}$ be the arc length reparameterization of $\varsigma(u^1) = (\alpha, \beta, \gamma)$ .Then, we may construct the dual curve $\hat{r} = r(u^1, \varepsilon) = r(u^1) + \varepsilon r^*(u^1)$ corresponding to the curve $r = r(u^1)$ Firstly, the dual part $r^*(u^1)$ of the dual curve is given from

$$r^*(u^1) = r(u^1) \times ON = (x_1, x_2, x_3) \times (0,0,1) = (-x_2, x_1, 0) \tag{3.6}$$

where O is the center of the sphere S^2 and N is the north pole. Then, the dual curve can be written in the following representation

$$\hat{r}(u^1) = r(u^1,\varepsilon) = (x_1, x_2, x_3) + \varepsilon(-x_2, x_1, 0) \tag{3.7}$$

Where $\varepsilon^2 = 0$ and $\hat{r} = r(u,\varepsilon)$ is a dual unite spherical curve corresponding to the plane curve C.

Construction of developable ruled surfaces

Here we construct a developable ruled surface corresponding to the dual unit space curve on $S2$ which corresponding to a plane curve. This ruled surface is given by

$$R(u^1,u^2) = r(u^1) \times r^*(u^1) + u^2 r(u^1) \tag{4.1}$$

where $r(u^1)$ and $r^*(u^1)$ are given by (3.5) and (3.6) respectively. In what follows, we construct developable ruled surfaces corresponding to plane curve of type Hypotrochoid , rose, ellipse, circle and catena

Dual developable ruled rose surfaces

Consider the Hypotrochoid curve given by

$$hy(u^1; n, m, c_1, c_2) = (x(u^1), y(u^1)) \subset C^2$$

where

$$x(u^1) = (n-m)\cos(u^1+\alpha) + c_1 m \cos\left(\tfrac{(n-m)u^1}{m} - c_2\right)$$
$$y(u^1) = (n-m)\sin(u^1+\alpha) - c_1 m \sin\left(\tfrac{(n-m)u^1}{m} - c_2\right) \tag{4.2}$$

Using the spherical representation in section(3) we have its corresponding dual curve and the constructed ruled surface in this case is called developable Hypotrochoid ruled surface. If we consider $n = 3,\ m = 1,\ c_1 = 2, c_2 = 0$ and using equation (4.2) its easy to see that

$$hy(u^1;,3,1,2,0) = (2\cos u^1 + 2\cos 2u^1, 2\sin u^1 - 2\sin 2u^1)$$

Then from (3.4) its easy to verify that the spherical image of this curve is given by

$$\zeta = (\alpha,\beta,\gamma) = M(2(\cos u^1 + \cos 2u^1), 2(\sin u^1 - \sin 2u^1), 4 + 4\cos 3u^1) \tag{4.3}$$

where $M(3 + 2\cos 3u^1) = 1$

Then we have the arc length reparamerization $r(u^1)$ of $\varsigma(u^1)$ as the following

$$r(u^1) = M(2(\cos u^1 + \cos 2u^1), 2(\sin u^1 - \sin 2u^1), 1 + 2\cos 3u^1) \tag{4.4}$$

Thus the dual part of the dual curve is

$$r^*(u^1) = r(u^1) \times (0,0,1) = M(-2(\sin u^1 - \sin 2u^1), 2(\cos u^1 + \cos 2u^1), 0) \tag{4.5}$$

and the dual unit spherical curve is given as

$$\hat{r}(u^1) = r^*(u^1) + \varepsilon r(u^1) \tag{4.6}$$

Then the developable ruled surface corresponding to this curve is defined as

$$R(u^1,u^2) = r(u^1) \times r^*(u^1) + u^2 r(u^1) \tag{4.7}$$

where

$$r(u^1) \times r^*(u^1) = M^2(M_1M_2, M_2M_3, M^2{}_1 + M^2{}_2 \tag{4.8}$$

where

$$M_1 = 2(\cos u^1 + \cos 2u^1, M_2 = \sin u^1 - \sin 2u^1$$
$$M_3 = 1 + 2\cos 3u^1 \tag{4.9}$$

The ruled surface defined by (4.4),(4.7) and (4.8) is called a developable ruled rose surface as shown in *Fig.*1,

Developable ruled ellipse surfaces

Consider the ellipse curve $C:(a\cos u^1, b\sin u^1, 0)$ and using the spherical representation (3.4) we have.

where $\upsilon(5+\cos u^1) = 1.$

Then the dual spherical image $\hat{r}(u^1) = r(u^1) + \varepsilon r^*(u^1)$ of the ellipse $(a = \sqrt{2}, b = 1)$ on S^2 is given from

$$r(u^1) = \nu(4\sqrt{2}\cos u^1, 4\sin u^1, -3 + \cos^2 u^1) \tag{4.11}$$

and its dual part is

$$r^*(u^1) = \nu(-4\sin u^1, 4\sqrt{2}\cos u^1, 0) \tag{4.12}$$

Thus, we have a developable ruled surface given by

$$R(u^1,u^2) = r(u^1) \times r^*(u^1) + u^2 r(u^1) \tag{4.13}$$

where

$$r(u^1) \times r^*(u^1) = \nu^2(4\sqrt{2}\cos u^1(-3+\cos^2 u^1), 4\sin u^1(-3+\cos^2 u^1), 16 + 16\cos^2 u^1) \tag{4.14}$$

This surface is called developable ruled ellipse surface as shown in Fig 2.

Developable ruled circle surfaces

Consider the unit circle curve $S:(\cos u^1, \sin u^1, 0)$ and using the same derivations as in the previous case we have the spherical image of the circle as in the following .

$$\zeta = \frac{4}{5}(\cos u^1, \sin u^1, 1) \tag{4.15}$$

Also the dual part $r^*(u^1)$ of the dual unite spherical curve $\hat{r}(u^1) = r(u^1) + \varepsilon r^*(u^1)$ is given by

$$r^*(u^1) = \frac{4}{5}(-\sin u^1, \cos u^1, 0) \tag{4.17}$$

Thus the corresponding developable ruled ellipse surface is given in the form

$$R(u^1,u^2) = r(u^1) \times r^*(u^1) + u^2 r(u^1) \tag{4.18}$$

where

$$r(u^1) \times r^*(u^1) = \frac{4}{25}(\cos u^1, -\sin u^1, 4) \tag{4.19}$$

This ruled surface plotted in Fig.3 and is called developable circle ruled surface

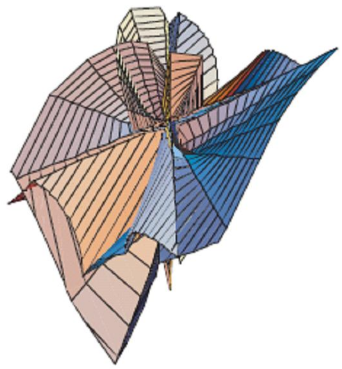

Figure 1: The developable ruled rose surface

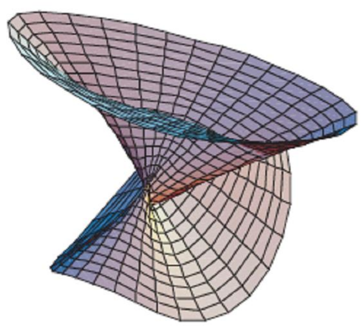

Figure 2: The developable ruled ellipse surface

Developable ruled catena surfaces

Consider the catena curve $C:(a\cosh\frac{u^1}{a}, u^1, 0)$ and using the same derivations as in the previous case we have the spherical image of the catena .The spherical image of the catena is given by (a=1)

$$\zeta = \eta(4\cosh u^1, u^1, 2(\cosh^2 u^1 + (u^1)^2)) \tag{4.20}$$

where $\eta(\cosh^2 u^1 + (u^1)^2 + 4) = 1$ and its unite parameterized curve is

$$r(u^1) = \eta(4\cosh u^1, u^1, \cosh^2 u^1 + (u^1)^2 - 4) \tag{4.21}$$

Also the dual part $r^*(u^1)$ of the dual unite spherical curve $\hat{r}(u^1) = r(u^1) + \varepsilon r^*(u^1)$ is

$$r^*(u^1) = \eta(-u^1, 4\cosh u^1, 0) \tag{4.22}$$

Then the developable ruled surface is represented by

$$R(u^1, u^2) = r(u^1) \times r^*(u^1) + u^2 r(u^1) \tag{4.23}$$

where

$$r(u^1) \times r^*(u^1) = \eta^2(4\cosh u^1(\cosh^2 u^1 + (u^1)^2 - 4), u^1(\cosh^2 u^1 + (u^1)^2 - 4), 16\cosh^2 u^1 + (u^1)^2) \tag{4.24}$$

This ruled surface plotted in Fig4 and is called developable catena. ruled surface

Corresponding author
Nassar H. Abdel-All
Mathematics Department, Faculty of Science, Assiut University Assiut 71516, Egypt
nhabdeal2002@yahoo.com

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