Structure and Some Geometric Properties of Generalized Cesáro Difference Sequence Space Defined by Weighted Means

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Abstract: In this paper, we define the sequence space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ to be consisting of all sequences $(x_k)_{k=0}^{\infty}$ for which $(x_k - x_{k-1})_{k=0}^{\infty}$ belongs to the sequence space $\ell[(a_n),(p_n),(q_n)]$ introduced by Altay and Başar [7]. We also define a modular functional on this space and show that it is a complete paranomed space, and when equipped with the Luxemburg norm is a Banach space, possessing H-property, and it is locally uniformly rotund (LUR) when $p_n > 1$, for all $n \in \mathbb{N}$. [Journal of American Science. 2010;6(10):19-24]. (ISSN: 1545-1003).

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Introduction

Let (X, ||.||) be a real Banach space and let B(X) (respe. S(X)) be the closed unit ball (resp. unit sphere) of X.

A point $x_0 \in S(X)$ is an H-point of B(X) if for any sequence (x_n) in X such that $\lim_{n\to\infty} \| x_n \| = 1$,the weak convergence of

x_n to x_0 $(write\ x_n \xrightarrow{W} x_0)$ implies

that $\lim_{n\to\infty} \| x_n - x_0 \| = 0$ .If every point of S(X) is an H-point of B(X); then X is said to have H–property (Kadec-Klee).Shortly; X is said to have the property (H), if for any sequence on the unit sphere of X, weak convergence coincides norm convergence.

A point $x \in S(X)$ is an extreme point of B(X), if for any $y, z \in S(X)$, the equality x= $\frac{y+z}{2}$ implies y=z.

A Banach space X is said to be Rotund (R) if for every point of S(X) is an extreme point of B(X).

A point $x \in S(X)$ is a locally uniformly rotund (LUR-point) if for any sequence (x_n) in B(X) such that $\lim_{n\to\infty} \| x_n + x \| = 2$, there holds

that $\lim_{n\to\infty} \| x_n - x \| = 0$, if every point of S(X) is a LUR-point of B(X), then X is called locally uniformly rotund (LUR). It is known if X is LUR, then it is (R) and posses property (H). However converse of this last statement is not true in general. Byω, we shall denote the space of all real or complex sequences and the set of natural numbers will denote by $\mathbb{N}$= {0,1,2,…}.

A linear topological space X over the real field $\mathbb{R}$ is said to be a

Paranormed space if there is a sub additive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$,

$g(-x) = g(x)$ and for any sequence (x_n) in X such that $\lim_{n\to\infty} g(x_n - x) = 0$, and any sequence (α_n) in $\mathbb{R}$ such that $\lim_{n\to\infty} |\alpha_n - \alpha| = 0$, we get $\lim_{n\to\infty} g(\alpha_n x_n - \alpha x) = 0$ .For these geometric

notions and their role in mathematics we refer to the monographs [1], [2], [3], [4], and [5]. Some of these geometric properties were studied for orlicz spaces in [9], [10], [11], and [12].

For a real vector space X, a function $\sigma : X \to [0,\infty]$ is called a modular, if it satisfies the following conditions:

(i) $\sigma(x) = 0 \Leftrightarrow x = 0$, $\forall x \in X$,

(ii) $\sigma(\lambda x) = \sigma(x)$, for all $\lambda \in \mathbb{R}$ with $|\lambda| = 1$,

(iii) $\sigma(\lambda x + \beta y) \le \sigma(x) + \sigma(y)$, $\forall x, y \in X$ $\forall \lambda, \beta \ge 0;\ \lambda + \beta = 1$.

Further, the modular σ is called convex if

(iv) $\sigma(\lambda x + \beta y) \le \lambda\sigma(x) + \beta\sigma(y)$, $\forall x, y \in X$, $\forall \lambda, \beta \ge 0;\ \lambda + \beta = 1$.By using the sequence space defined in [6], Altay and Başar [7] defined the sequence space $\ell[(a_n),(p_n),(q_n)]$ as $\ell[(a_n),(p_n),(q_n)] =$ $\left\{ x \in \omega : \sum_{n=0}^{\infty} (a_n \sum_{k=0}^{n} q_k |x_k|)^{p_n} < \infty \right\}$. They also showed that the space $\ell[(a_n),(p_n),(q_n)]$ is a complete linear metric space paranormed by $g(x) = \left[\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k \,|\, x_k \,|\right)^{p_n} \right]^{\frac{1}{M}}$ also V.Karakaya and N.Şimşek [8] proved that this space is a Banach space and posses Kadec-Klee (H).

The idea of difference sequence was first introduced by Kizmaz [14]. Write $\Delta x_k = x_k - x_{k-1}$ for all $k \in \mathbb{N}$ and $\Delta : \omega \to \omega$ be the difference operator defined by

$$\Delta x = (x_k - x_{k-1})_{k=0}^{\infty}, \text{ with } x_{-1} = 0.$$

We now introduce a generalized modular difference sequence space defined by weighted means

Definition: let $(a_n),(q_n)$ and (p_n) be sequences of positive real numbers we define the space $\ell_\Delta((a_n),(p_n),(q_n)) = \{x \in \omega : \sigma(\lambda x) < \infty,\ \exists \lambda > 0\}$,

where $\sigma(x) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} q_k \,|\, \Delta x_k \,|\right)^{P_n}$.And the Luxemburg norm on the sequence space $\ell_\Delta[(a_n),(p_n),(q_n)]$ is defined as follows: $\| x \| = \inf\left\{ \lambda > 0 : \sigma(\frac{x}{\lambda}) \le 1 \right\}$,

$\forall x \in \ell_\Delta((a_n),(p_n),(q_n))$. In the case when the sequence (p_n) is bounded we can simply write $\ell_\Delta((a_n),(p_n),(q_n)) =$

$$\left\{ x \in \omega : \sum_{n=0}^{\infty} (a_n \sum_{k=0}^{n} q_k |\Delta x_k|)^{p_n} < \infty \right\}.$$

Throughout this paper, the sequence (p_n) is considered to be bounded with $p_n > 1\ \forall\ n \in \mathbb{N}$ and let $\sup_r p_r = H$. For any bounded sequence of positive numbers (p_k), we have

$$|\, a_k + b_k \,|^{p_k} \le 2^{H-1}(|\, a_k \,|^{p_k} + |\, b_k \,|^{p_k}),$$

where $p_k \ge 1\ \forall\ k \in \mathbb{N}$.

Lemma (1):

The functional σ is convex modular on $\ell_\Delta[(a_n),(p_n),(q_n)]$.

Proof: It can be proved with standard techniques in a similar way as in [5, 15]

Lemma (2):

For any $x \in \ell_\Delta[(a_n),(p_n),(q_n)]$, the functional σ on $\ell_\Delta[(a_n),(p_n),(q_n)]$ satisfies the following properties:

(i) If 0<r<1, then $r^H \sigma\left(\frac{x}{r}\right) \le \sigma(x)$ and $\sigma(rx) \le r\sigma(x)$.

(ii) If r>1, then $\sigma(x) \le r^H \sigma\left(\frac{x}{r}\right)$.

(iii) If r≥1, then $\sigma(x) \le r\sigma(x) \le \sigma(rx)$.

Proof: It can be proved with standard techniques in a similar way as in [5, 15].

Lemma (3):

For any $x \in \ell_\Delta[(a_n),(p_n),(q_n)]$, the following assertions are satisfied:

(i) If $\| x \| < 1$, then $\sigma(x) \le \| x \|$,

(ii) if $\| x \| > 1$, then $\sigma(x) \ge \| x \|$,

(iii) $\| x \| = 1$ if and only if $\sigma(x) = 1$,

(iv) if $0 < r < 1$ and $\| x \| > r$, then $\sigma(x) > r^H$,

(v) if $r \ge 1$ and $\| x \| < r$, then $\sigma(x) < r^H$.

Proof: It can be proved with standard techniques in a similar way as in [5, 15].

Lemma (4):

Let (x_n) be a sequence in $\ell_\Delta[(a_n),(p_n),(q_n)]$,

(i) if $\lim_{n\to\infty} \| x_n \| = 1$, then $\lim_{n\to\infty} \sigma(x_n) = 1$,

(ii) if $\lim_{n\to\infty} \sigma(x_n) = 0$, then $\lim_{n\to\infty} \| x_n \| = 0$.

Proof:

(i) Suppose that $\lim_{n\to\infty} \| x_n \| = 1$. Then for any $\varepsilon \in (0,1)$ there exists n_o such that $1-\varepsilon < \| x_n \| < 1+\varepsilon \ \forall\ n \ge n_0$. By lemma (3), $(1-\varepsilon)^H < \sigma(x_n) < (1+\varepsilon)^H$ implies that $\lim_{n\to\infty} \sigma(x_n) = 1$.

(ii) If $\lim_{n\to\infty} \| x_n \| \ne 0$, then there is an $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) such that

$\| x_{n_k} \| > \varepsilon^H \ \forall\ k \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \sigma(x_{n_k}) \ne 0$ and hence $\lim_{n\to\infty} \sigma(x_n) \ne 0$.

Main results

Theorem (1): $\ell_\Delta[(a_n),(p_n),(q_n)]$ is a Banach space with respect to the Luxemburg norm defined by $\| x \| = \inf\left\{ \rho > 0 : \sigma\left(\frac{x}{\rho}\right) \le 1 \right\}$.

Proof: Let $x_n = (x_n(k))_{k=o}^{\infty}, n = 0,1,2,...$ be a Cauchy sequence in $\ell_\Delta[(a_n),(p_n),(q_n)]$ according to the Luxemburg norm. Thus $\forall\, \varepsilon \in (0,1)\ \exists\ n_o$ such that $\| x_n - x_m \| < \varepsilon^H \ \forall\ m,n \ge n_0$. By the lemma 3(i) we obtain.

$$\sigma(x^n - x^m) < \| x_n - x_m \| < \varepsilon^H, \quad (1)$$

$\forall\, m,n \ge n_0$. That is

$$\sum_{r=0}^{\infty}\left(a_r \sum_{k=0}^{r} q_k \,|\Delta x_n(k) - \Delta x_m(k)|\right)^{P_r} < \varepsilon^H$$

$\forall\, m,n \ge n_0$. For fixed k we get that $|\Delta x_n(k) - \Delta x_m(k)| < \varepsilon$ and the sequence $(\Delta x_n(k))$ is a Cauchy sequence of real numbers. Let $\Delta x(k) = \lim_{n\to\infty} \Delta x_n(k)$, then from inequality (1), we can write

$$\sum_{r=0}^{\infty}\left(a_n \sum_{k=0}^{r} q_k \,|\Delta x_n(k) - \Delta x(k)|\right)^{P_r} < \varepsilon^H,$$

$\forall\, n \ge n_0$.

That is, $\sigma(x_n - x) < \varepsilon^H \Rightarrow \lim_{n\to\infty}(x_n) = x$.

By the following calculations,

$$\sum_{r=0}^{\infty}\left(a_r \sum_{k=0}^{r} q_k \,|\Delta x(k)|\right)^{P_r} = \sum_{r=0}^{\infty}\left(a_r \left(\sum_{k=0}^{r} q_k \,|\Delta x(k) - \Delta x_n(k)| + \sum_{k=0}^{r} q_k \,|\Delta x_n(k)|\right)\right)^{P_r}$$

$$\le 2^{H-1}\left[\left(\sum_{r=0}^{\infty}\left(a_r \sum_{k=0}^{r} q_k \,|\Delta x(k) - \Delta x_n(k)|\right)^{P_r}\right) + \sum_{r=0}^{\infty}\left(a_r \sum_{k=0}^{r} q_k \,|\Delta x_n(k)|\right)^{P_r}\right]$$

$< \varepsilon$.

we see that the sequence x_n converges to $x = (x(k)) \in \ell_\Delta[(a_n),(p_n),(q_n)]$. This completes the proof.

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Theorem(2) Let $(x_n) \subseteq B(\ell(p))$ and $(y_n) \subseteq B(\ell(p))$. If $\lim_{n\to\infty} \sigma\left(\frac{x_n + y_n}{2}\right) = 1$, then $\lim_{n\to\infty}(x_n(k) - y_n(k)) = 0$, for all $k \in \mathbb{N}$.

Proof: See [15Proposition 2.6]

Theorem(3): Let $x_m \in B(\ell_\Delta[(a_n),(p_n),(q_n)], \forall\, m \in \mathbb{N}$.

If $\underset{n\to\infty}{Lim}\,\sigma\left(\frac{x_n+x}{2}\right)=1$, then $\underset{n\to\infty}{Lim}\,x_n(k)=x(k)\,,\forall\; k\in\mathbb{N}$.

Proof:

For each m and $k\in\mathbb{N}$, let

$$S_k^m=\begin{cases}\operatorname{sgn}(\Delta x_m(k)+\Delta x(k)) & \text{If } \Delta x_m(k)+\Delta x(k)\neq 0\\ 1 & \text{If } \Delta x_m(k)+\Delta x(k)=0\end{cases}$$

.

Hence we have $1=\underset{m\to\infty}{Lim}\,\sigma\left(\frac{x_m+x}{2}\right)=$

$$=\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k\left|\frac{\Delta x_m(k)+\Delta x(k)}{2}\right|\right)^{P_n}=$$

$$\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_kS_k^m.\frac{\Delta x_m(k)}{2}+a_n\sum_{k=0}^{n}q_kS_k^m.\frac{\Delta x(k)}{2}\right)^{P_n}$$

. (2)

Let $\alpha_n^m=a_n\sum_{k=0}^{n}q_kS_k^m.\Delta x_m(k)$

and $\beta_n^m=a_n\sum_{k=0}^{n}q_kS_k^m.\Delta x(k)$

$\forall\, m,n\in\mathbb{N}$,

then $(\alpha^m),(\beta^m)\in\ell_\Delta[(a_n),(p_n),(q_n)$ and from (2) we have

$\underset{m\to\infty}{Lim}\,\sigma\left(\frac{\alpha^m+\beta^m}{2}\right)=1$ .From Theorem (4) we have

$$\underset{n\to\infty}{Lim}\left(\alpha_k^m-\beta_k^m\right)=0\;,\forall\, k\in\mathbb{N}. \qquad (3)$$

Now we shall prove that $\underset{m\to\infty}{Lim}\,x_m(k)=x(k)\,,\forall\, k\in\mathbb{N}$. From (3) at $k=0$ we have $\underset{m\to\infty}{Lim}\left(s_0^m\Delta x_m(0)-s_0^m\Delta x_m(0)\right)=0$ .This implies that $\underset{m\to\infty}{Lim}\,x_m(0)=x(0)$.

Assume that $\underset{m\to\infty}{Lim}\,x_m(k)=x(k)\,,\forall\, k\le n-1$. Then we have

$\underset{m\to\infty}{Lim}\,s_k^m(x_m(k)-x(k))=0$, for all $k\le n-1$

$$s_n^mq_n(\Delta x_m(k)-\Delta x(k))=\frac{1}{a_n}\left(\alpha_n^m-\beta_n^m\right)-\sum_{k=0}^{n-1}q_ks_k^m(\Delta x_m(k)-\Delta x(k))$$

. (4)

It follows that from (3) and (4) that

$\underset{m\to\infty}{Lim}\,s_n^mq_n(\Delta x_m(k)-\Delta x(k))\to 0$ .This implies $\underset{m\to\infty}{Lim}\,x_m(k)=x(k)$, $\forall\; k\in\mathbb{N}$.

Theorem (4): The space $\ell_\Delta[(a_n),(p_n),(q_n)]$ is LUR.

Proof:

Let $(x_n)\subseteq B(\ell_\Delta[(a_n),(p_n),(q_n)])$ and $x\in S(\ell_\Delta[(a_n),(p_n),(q_n)])$ be such that $\underset{n\to\infty}{Lim}\left\|\frac{x_n+x}{2}\right\|=1$ .By lemma (4-i) we have $\underset{m\to\infty}{Lim}\,\sigma(\frac{x_n+x}{2})=1$ .Since

$\sigma(x)=\sum_{r=0}^{\infty}(a_r\sum_{k=0}^{r}q_k\,|\,\Delta x(k)\,|)^{P_r}<\infty$, then for $\varepsilon>0$, there exists $r_0\in\mathbb{N}$ such that

$$\sum_{r=r_0+1}^{\infty}(a_r\sum_{k=0}^{r}q_k\,|\,\Delta x(k)\,|)^{P_r}<\frac{\varepsilon}{3(2^{H+1})} \qquad (5)$$

Since

$$\underset{n\to\infty}{Lim}\left(\sigma(x_n)-\sum_{r=0}^{r_0}(a_r\sum_{k=0}^{r}q_k\,|\,\Delta x_n(k)\,|)^{P_r}\right)=\sigma(x)-\sum_{r=0}^{r_0}(a_r\sum_{k=0}^{r}q_k\,|\,\Delta x(k)\,|)^{P_r}$$

, and $\underset{n\to\infty}{Lim}\,x_n(k)=x(k)\;\forall\, k\in\mathbb{N}$, $\exists\, n_0\in\mathbb{N}$ such that

$$\left|\sum_{r=r_0+1}^{\infty}(a_r\sum_{k=0}^{r}q_k\,|\,\Delta x_n(k)\,|)^{P_r}-\sum_{r=r_0+1}^{\infty}(a_r\sum_{k=0}^{r}q_k\,|\,\Delta x(k)\,|)^{P_r}\right|<\frac{\varepsilon}{3(2^H)} \qquad (6)$$

$\forall\, n\ge n_0$, since $\underset{n\to\infty}{Lim}\,x_n(k)=x(k)$, since Δ is

Continuous operator Hence $\forall\, n\ge n_0$ we have $|\,\Delta x_n(k)-\Delta x(k)\,|<\varepsilon$ .As a result$\forall$ $n\ge n_0$, we get

$$\sum_{r=0}^{r_0}(a_r\sum_{k=0}^{r}q_k\,|\Delta x_n(k)-\Delta x(k)\,|)^{P_r}<\frac{\varepsilon}{3}. \quad (7)$$

Then from (5), (6) and(7)it follows that $\forall\, n\geq n_0$,we have

$$\sigma(x_n-x)=\sum_{r=0}^{\infty}\left(a_r\sum_{k=0}^{r}q_k\,|\Delta x_n(k)-\Delta x(k)\,|\right)^{P_r}$$

$$=\sum_{r=0}^{r_0}(a_r\sum_{k=0}^{r}q_k\,|\Delta x_n(k)-\Delta x(k)|)^{P_r}+\sum_{r=r_0+1}^{\infty}(a_r\sum_{k=0}^{r}q_k\,|\Delta x_n(k)-\Delta x(k)|)^{P_r}$$

$$<\frac{\varepsilon}{3}+2^H\left[\sum_{r=r_0+1}^{\infty}\left(a_r\sum_{k=0}^{r}q_k\,|\Delta x_n(k)|\right)^{P_r}+\sum_{r=r_0+1}^{\infty}\left(a_r\sum_{k=0}^{r}q_k\,|\Delta x(k)|\right)^{P_r}\right]$$

$$<\frac{\varepsilon}{3}+2^H\left[2\sum_{r=r_0+1}^{\infty}\left(a_r\sum_{k=0}^{r}q_k\,|\Delta x(k)|\right)^{P_r}+\frac{\varepsilon}{3(2^H)}\right]$$

$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

This shows that $\lim\limits_{n\to\infty}\sigma(x_n-x)=0$. Hence by lemma 4 (ii), we have $\lim\limits_{n\to\infty}\|x_n-x\|=0$, the space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ is LUR.

Theorem (5): The space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ is a complete linear metric space with respect to the paranorm defined by

$$g(x)=\left[\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k\,|\Delta x_k\,|\right)^{p_n}\right]^{\frac{1}{H}}.$$

Proof: The proof of linearity of $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ with respect to the coordinate wise addition and multiplication follows from the following inequalities which are satisfied for all $x,y\in\ell_{\Delta}[(a_n),(p_n),(q_n)]$

$$\left[\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|\Delta x_k+\Delta y_k|\right)^{p_n}\right]^{\frac{1}{H}}\leq$$

$$\left[\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k\,|\Delta x_k|\right)^{p_n}\right]^{\frac{1}{H}}+$$

$$+\left[\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k\,|\Delta y_k|\right)^{p_n}\right]^{\frac{1}{H}} \quad (8),$$

and $|\alpha|^{p_n}\leq\max\{1,|\alpha|^H\}$ for any $\alpha\in\mathbb{R}$. We now verify that g(x) is a paranorm over the space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$.In fact,

(i) $g(\theta)=0$ (Clearly),

(ii) $g(-x)=g(x)$, $\forall x\in\ell_{\Delta}[(a_n),(p_n),(q_n)]$,

(iii) $g(x+y)\leq g(x)+g(y)$, $\forall x,y\in\ell_{\Delta}[(a_n),(p_n),(q_n)]$, follows from the inequality (8).

(iv)Let (x_m) be any sequence in $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ such that $\lim\limits_{n\to\infty}g(x_m-x)=0$; let (α_m) be any sequence in $\mathbb{R}$ such that $\lim\limits_{m\to\infty}|\alpha_m-\alpha|=0$, since $x_m=x+(x_m-x)$ then we get $g(x_m)\leq g(x)+g(x_m-x)$.

Hence $\{g(x_m)\}$ is bounded and we have

$$g(\alpha_m x_m-\alpha x)=\left[\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|\alpha_m\Delta x_m(k)-\alpha\Delta x(k)|\right)^{p_n}\right]^{\frac{1}{H}}=$$

$$\left[\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}q_k|(\alpha_m-\alpha)(\Delta x_m(k))+\alpha(\Delta x_m(k)-\Delta x(k))|\right)^{p_n}\right]^{\frac{1}{H}}$$

, this tends to zero as $m\to\infty$.

The completence of the space $\ell_{\Delta}[(a_n),(p_n),(q_n)]$ is a routine verification by using standard techniques as theorem (1).

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