Calculation of Creeping Flow Past a Sphere Using Direct Boundary Element Method

Ghulam Muhammad*, Nawazish Ali Shah

Department of Mathematics, University of Engineering & Technology Lahore – 54890, Pakistan

Corresponding Author, e-mail: chgm2004@yahoo.com

Abstract:

In this paper, a steady, incompressible creeping flow past a sphere is calculated using direct boundary element method (DBEM). The surface of the sphere is discretized into quadrilateral elements over which the velocity distribution is calculated. The computed results are compared with analytical results. It is found that both these results are in good agreement. [Journal of American Science 2010;6(6):162-165], (ISSN: 1545-1003).

Keywords:

Boundary element method , Creeping flow past a sphere.

Introduction

In recent past, the well-known computational methods such as finite difference method (FDM), finite element method (FEM) and boundary element method (BEM) have been applied for the flow field calculations around objects and in such methods, the whole region of flow field is discretised. Whereas in boundary element method only the surface of the body under consideration is discretized into different types of boundary elements (C.A.Brebbia & S.Walker,1980). BEM is well–suited to two-and three-dimensional problems for which finite elements are not suitable or insufficient, especially for problems where domain is exterior to the boundary, as in the case of flow past bodies. The most important features of BEM are the much smaller system of equations and considerable reduction in data, which are essential to run a computer program efficiently. That is why; BEM is more accurate, efficient and economical than other competitive computational methods. Boundary element methods are further classified into direct and indirect methods. In the past, indirect method has been applied to calculate potential flow around arbitrary bodies (Hess & Smith, 1967, Mushtaq, 2010). In present paper, DBEM is applied to calculate creeping flow past a sphere. The study of flow past a sphere is of great practical importance in fluid dynamics. In creeping flow, the inertial effects become very small, whereas, the viscous effects become dominant. Therefore, the steady flow Navier-Stokes' equations are greatly simplified by neglecting the inertia terms (J.F. Dougles, J.. Gasiorek & J. A. Swaffield,1990). The direct boundary element method (DBEM) for potential flow calculations around objects was applied first in the past by Morino (1975). In recent past, the direct element method has been applied by the author for flow field calculations around two- and three-dimensional bodies.

Mathematical Formulation of Steady and Incompressible Creeping Flow

The differential equations governing the creeping flow are the continuity equation and the Navier – Stokes' equations (Milne–Thomson 1968)

$$\nabla \cdot \vec{V} = 0 \qquad (1)$$

and

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V} \qquad (2)$$

In the case of very creeping motion or in the case of very highly viscous fluid , the Reynold's number will be small $(\mathrm{Re} << 1)$. In such cases the inertia term or convective acceleration term $\left(\vec{V} \cdot \nabla \right) \vec{V}$ is approximately zero. Thus equations (1) and (2) reduce to

$$\nabla \cdot \vec{V} = 0 \qquad (3)$$

and

$$\frac{\partial \vec{V}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V} \qquad (4)$$

These equations are known as Stokes' equations for very creeping motion . Flows which satisfy equation (4) are called creeping flows .

Equation (4) represents the following three scalar equations .

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u \\ \frac{\partial v}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v \\ \frac{\partial w}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w \end{aligned}\right] \qquad (5)$$

These equations together with the continuity equation (1) represent four scalar equations in four unknown u , v , w , and p . The great simplification in Stokes' equations is that these equations are now linear . In the case of steady flow , Stokes' equation (4) reduces to

$$\nabla p = \mu \nabla^2 \vec{V} \qquad (6)$$

Equation (6) can be written in scalar form as

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \mu \nabla^2 u \\ \frac{\partial p}{\partial y} &= \mu \nabla^2 v \\ \frac{\partial p}{\partial z} &= \mu \nabla^2 w \end{aligned}\right] \qquad (7)$$

The Stokes' equations are considerably simple from mathematical point of view as they are linear differential equations . Moreover , their order remain the same as that of full Navier – Stokes' equations so that as many boundary conditions may be satisfied with the Stokes' equations as with the full Navier – Stokes' equations .

The Navier-Stoke equations for creeping incompressible viscous flow in the absence of body force is as follows:

$$\frac{\partial \vec{V}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} \qquad (8)$$

To obtain the equation governing the pressure , take the divergence of both sides of equation (8), we get

$$\nabla \cdot \frac{\partial \vec{V}}{\partial t} = -\frac{1}{\rho}\nabla \cdot \nabla p + \nu \nabla \cdot \nabla^2 \vec{V} \qquad (9)$$

or $$\frac{\partial}{\partial t}(\nabla \cdot \vec{V}) = -\frac{1}{\rho}\nabla^2 p + \nu \nabla^2 (\nabla \cdot \vec{V}) \qquad (10)$$

Using continuity equation (1) , equation (10) becomes

$$\nabla^2 p = 0 \qquad (11)$$

i.e. for very slow motion the pressure p satisfies Laplace's equation and is therefore a harmonic function .

Steady Creeping Flow Past a Sphere

This problem was first solved by Stokes' and is often referred to as **Stokes' flow** or **Stokes' law** . Stoke was the first who analytically solved the problem of creeping flow.

Let a solid sphere of radius 'a' be held fixed in a uniform stream U flowing steadily in the positive direction of the z – axis. Let the centre of the sphere be the origin of the coordinate system. Let z – axis be in the direction of the uniform stream in the coordinate system, as shown in figure (1). The streamlines are symmetrical around the sphere; therefore there is no wake on the rear of a sphere. The flow past a sphere varies with the Reynolds number. In general, the larger the Reynolds number, the smaller the region of flow field in which the viscous effects are paramount and vice versa.

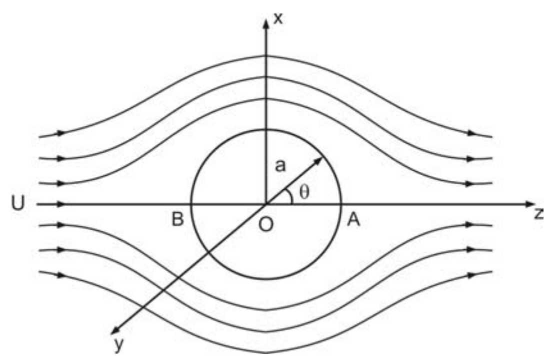


Figure (1)

Stream Function for Creeping Flow

$$\psi = -\frac{1}{4}\frac{U a^3}{r}\sin^2\theta + \frac{3}{4} U a r \sin^2\theta - \frac{1}{2} U r^2 \sin^2\theta$$

$$= \frac{3}{4} U a r \left(1 - \frac{1}{3}\frac{a^2}{r^2} - \frac{2}{3}\frac{r}{a}\right)\sin^2\theta \qquad (12)$$

Velocity Distribution

The velocity components in terms of Stokes' stream function are (Milne–Thomson 1968, White 1991)

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \text{ and } v_\theta = \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r} \qquad (13)$$

The velocity components in this case are

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta} = U\left(1-\frac{3a}{2r}+\frac{a^3}{2r^3}\right)\cos\theta$$

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = U\left(-1+\frac{3a}{4r}+\frac{a^3}{4r^3}\right)\sin\theta$$

$$V = \sqrt{v_r^2+v_\theta^2} = U\sqrt{\left(1-\frac{3a}{2r}+\frac{a^3}{2r^3}\right)^2\cos^2\theta+\left(-1+\frac{3a}{4r}+\frac{a^3}{4r^3}\right)^2\sin^2\theta} \quad (13)$$

The boundary conditions which must be satisfied by the flow are

$$v_r = 0,\quad v_\theta = 0 \quad \text{at} \quad r = a$$

$$\text{and } \psi = -\frac{1}{2}U r^2 \sin^2\theta \quad \text{at} \quad r = \infty .$$

Equation of DBEM:

For three-dimensional exterior flow problems, the equation of direct boundary element method over the surface 'S'of the body is given by

$$c_i\phi_i = \phi_\infty - \frac{1}{4\pi}\iint_S \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS + \frac{1}{4\pi}\iint_{S-i}\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \quad (14)$$

Discretization of Sphere:

The surface of the sphere is discretized into quadrilateral elements. The scheme of discretization is as shown in the figure (2).

The direct boundary element method is applied to calculate the creeping flow solution around the sphere for which the analytical solution is available

Consider the surface of the sphere in one octant to be divided into three quadrilateral elements by joining the centroid of the surface with the mid points of the curves in the coordinate planes as shown in figure (2) (Mustaq et al, 2009).

Then each element is divided further into four elements by joining the centroid of that element with the mid–point of each side of the element. Thus one octant of the surface of the sphere is divided into 12 elements and the whole surface of the body is divided into 96 boundary elements. The above mentioned method is adopted in order to produce a uniform distribution of element over the surface of the body .

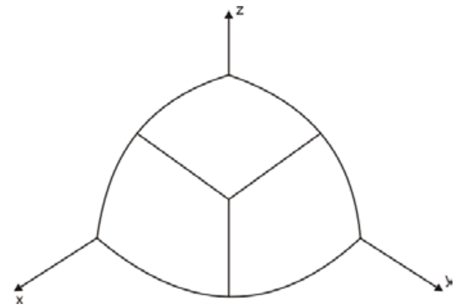


Figure (2)

Figure (3) shows the method for finding the coordinate (x_p, y_p, z_p) of any point P on the surface of the sphere.

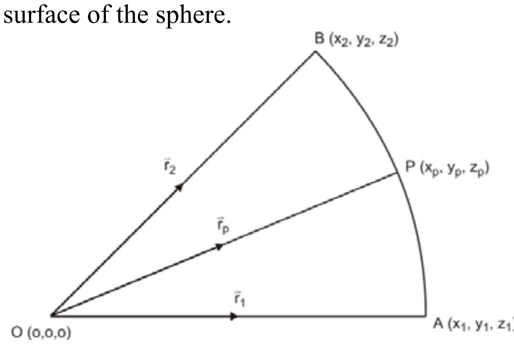


Figure (3)

From above figure, we have the following equation

$$|\vec{r}_p| = 1$$

$$\vec{r}_p \cdot \vec{r}_1 = \vec{r}_p \cdot \vec{r}_2$$

$$(\vec{r}_1 - \vec{r}_2)\cdot\vec{r}_p = 0$$

or in cartesian form

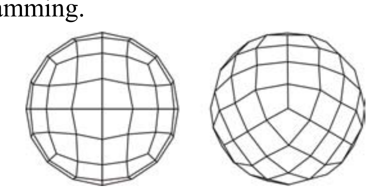
$$x_p^2 + y_p^2 + z_p^2 = 1$$

$$x_p(x_1 - x_2) + y_p(y_1 - y_2) + z_p(z_1 - z_2) = 0$$

$$x_p(y_1 z_2 - z_1 y_2) + y_p(x_2 z_1 - x_1 z_2) + z_p(x_1 y_2 - x_2 y_1) = 0$$

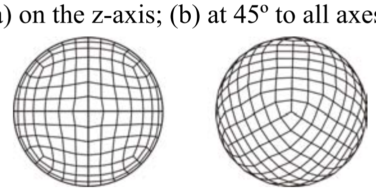
As the body possesses planes of symmetry, this fact may be used in the input to the program and only the non–redundant portion need be specified by input points. The other portions are automatically taken into account. The planes of symmetry are taken to be the coordinate planes of the reference coordinate system. The advantage of the use of symmetry is that it reduces the order of the resulting system of equations and consequently reduces the computing time in running a program. As a sphere is symmetric with respect to all three coordinate planes of the reference coordinate system, only one eighth of the body surface need be specified by the input points, while the other seven–eighth can be accounted for by symmetry.

The sphere is discretised into 96 and 384 boundary elements and the computed velocity distributions are compared with analytical solutions for the sphere using Fortran programming.



(a) (b)

Figure (4): Discretization of sphere into 96 boundary elements. The point of observation is (a) on the z-axis; (b) at 45º to all axes.



(a) (b)

Figure (5): Discretization of sphere into 384 boundary elements. The point of observation is (a) on the z-axis; (b) at 45º to all axes.

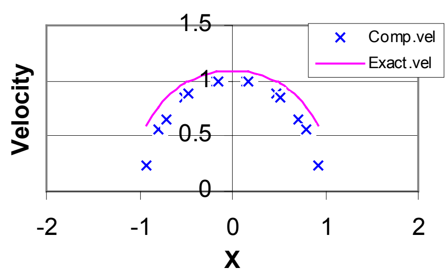


Figure (6): Comparison of computed and analytical velocity distributions over the surface of the sphere using 96 boundary elements.

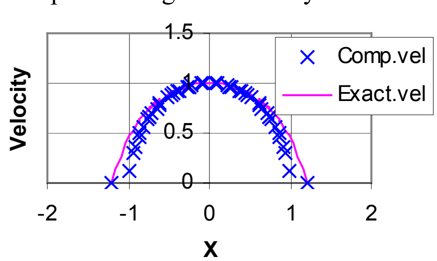


Figure (7): Comparison of computed and analytical velocity distributions over the surface of the sphere using 384 boundary elements.

Since the streamlines are symmetrical around the sphere, the graphs shown above are symmetrical on both sides. At the top of figure (7), the computed results are convergent with the exact results and as we come down, the computed results are slightly different with the analytical ones due to increase of viscous effects.

Conclusion:

Direct boundary element method has been used to calculate slow flow past a sphere using different number of boundary elements. The computed velocities obtained in this way are compared with exact velocities for this flow over the boundary of the sphere. From the above figures, it is concluded that the computed values are in good agreement with the exact values for the body of the sphere..

References:

[1] Brebbia, C.A. and Walker, S.: “ Boundary Element Techniques in Engineering” , Newnes-Butterworths, 1980.

[2] J.F. Douglas, J.M. Gasiorek & J.A. Swaffield.: “ Fluid Mechanics”, Longman Group,Ltd U.K.,1990.

[3] Shah, N.A.: “The Boundary Element Method for road vehicle aerodynamics”, PhD. Thesis (1985), Department of Engineering Mathematics, Loughborough University of Technology, U.K.

[4] Milne–Thomson, L.M.: “Theoretical Hydrodynamics”, 5th Edition, London Macmillan & Co. Ltd., (1968).

[5] Hess, J.L. and Smith, A.M.O.: “Calculation of potential flow about arbitrary bodies”, Progress in Aeronautical Sciences, Pergamon Press 1967, 8: 1–158.

[6] Morino L., Chen, Lee-Tzong and Suciu, E.O.: “A steady and oscillatory subsonic and supersonic aerodynamics around complex configuration”, AIAA Journal, 13,1975, 368-374.

[7] Mushtaq, M., & Shah, N.A.: “Indirect Boundary Element Method for Calculation of Compressible Flow Past a Symmetric Aerofoil with Constant Element Approach”, Journal of American Science, Vol. 6, No. 5 (2010) U.S.A.

[8] Mushtaq, M. , Shah, N.A & Muhammad, G : Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Sphere, Krag J. of Sc., 2009,31,25-32.

[9] Frank M White: “Viscous Fluid Flow”, 2nd Edition McGraw–Hill, Inc., 1991.